
SIGILLVM ACADEMIÆ HARVARDIANÆ IN NOV: ANG:
CHRISTO ET ECCLESIÆ

8

THE QUARTERLY JOURNAL OF PURE AND APPLIED MATHEMATICS.

CAMBRIDGE:
Printed by W. Metcalfe, Trinity Street, Corner of Green Street.

THE QUARTERLY JOURNAL OF PURE AND APPLIED MATHEMATICS.

EDITED BY

J. J. SYLVESTER, M.A., F.R.S.,
PROFESSOR OF MATHEMATICS IN THE ROYAL MILITARY ACADEMY, WOOLWICH; AND

N. M. FERRERS, M.A.,
FELLOW OF GONVILLE AND CAIUS COLLEGE, CAMBRIDGE:

ASSISTED BY

G. G. STOKES, M.A., F.R.S.,
LUCASIAN PROFESSOR OF MATHEMATICS IN THE UNIVERSITY OF CAMBRIDGE;

A. CAYLEY, M.A., F.R.S.,
SADLERIAN PROFESSOR OF MATHEMATICS IN THE UNIVERSITY OF CAMBRIDGE;

M. HERMITE,
CORRESPONDING EDITOR IN PARIS.

VOL. VIII.

ὅ τι οὐσία πρὸς γένεσιν, ἐπιστημὴ πρὸς πίστιν καὶ διάνοια πρὸς εἰκασίαν ἐστι.

LONDON:
LONGMANS, GREEN, AND CO.,
PATERNOSTER ROW.

1867.

1871, July 1.
Haven Fund.

CONTENTS OF VOLUME VIII.

No. 29. June, 1866.

THE

QUARTERLY JOURNAL

OF

PURE AND APPLIED

MATHEMATICS.

EDITED BY

J. J. SYLVESTER, M.A., F.R.S.,

PROFESSOR OF MATHEMATICS IN THE ROYAL MILITARY ACADEMY, WOOLWICH; AND

N. M. FERRERS, M.A.,

FELLOW AND TUTOR OF GONVILLE AND CAIUS COLLEGE, CAMBRIDGE:

ASSISTED BY

G. G. STOKES, M.A., F.R.S.,

LUCASIAN PROFESSOR OF MATHEMATICS IN THE UNIVERSITY OF CAMBRIDGE;

A. CAYLEY, M.A., F.R.S.,

SADLERIAN PROFESSOR OF MATHEMATICS IN THE UNIVERSITY OF CAMBRIDGE; AND

M. HERMITE,

CORRESPONDING EDITOR IN PARIS.

ὅ τι οὐσία πρὸς γένεσιν, ἐπιστήμη πρὸς πίστιν καὶ διάνοια πρὸς εἰκασίαν ἐστι.

LONDON:

LONGMANS, GREEN, AND CO.,

PATERNOSTER ROW.

1866.

W. METCALFE, PRINTER, GREEN STREET, CAMBRIDGE.

PRICE FIVE SHILLINGS.

CONTENTS.

NOTICES TO CORRESPONDENTS.

The following Papers have been received:

Mr. Walton, "On Certain Transformations in the Calculus of Operations;" "A Demonstration of Fourier's Theorem;" "On the Symbol of Operation $x\dfrac{d}{dx}$;" and "On the Debility of Large Trees and Animals."

Professor Cayley, "Specimen Table $M \equiv a^\alpha b^\beta$ (mod. N) for any Prime or Composite Modulus;" "Tables of the Binary Cubic Forms for the Negative Determinants, $\equiv 0$ (Mod. 4), from -4 to -400; and $\equiv 1$ (Mod. 4), from -3 to -99; and for five Irregular Negative Determinants;" "Theorem relating to the Four Conics which touch the same Two Lines, and pass through the same Three Points;" "On the Conics which pass through two given Points, and touch two given Lines;" "Solution of a Problem of Elimination;" and "On a certain Envelope depending on a Triangle Inscribed in a Circle."

Mr. W. F. Walker, "Three brief Notes on Questions in Analytical Geometry."

Mr. J. Power, "On the Problem of the Fifteen School Girls."

Mr. Joseph Horner, "Notes on Determinants."

Worontzof, "On the Generalization of Certain Formulæ investigated by Mr. Blissard."

Mr. Purser, "On a Theorem in Quadrics."

Mr. Ellis, "Investigation of an Algebraical Formula."

Herr. L. Schläfli, "A consequence of Mr. Cayley's Theory of Skew Determinants, concerning the Displacement of a Rigid System of an Even Number of Dimensions about a Fixed Origin."

Mr. C. Taylor, "On some special Forms of Conics."

Mr. E. J. Routh, "On Tangential Coordinates."

Notice to Subscribers.—The first Seven Volumes of the 'Journal,' bound in calf, gilt, or any of the back Numbers, can be had of the Publishers, or at the Printing Office, Green Street, Cambridge.

No. 30 will be published in October, 1866.

THE

QUARTERLY JOURNAL

OF

PURE AND APPLIED MATHEMATICS.

ON THE NUMBER OF SURFACES OF THE SECOND DEGREE WHICH CAN BE DESCRIBED TO SATISFY NINE CONDITIONS.

By the Rev. GEORGE SALMON.

1. We use the notation $N(r, s, t)$ to denote the number of surfaces of the second degree which can be described to pass through r points, to touch s lines, and to touch t planes; and it is desired to determine this number for all values of r, s, t, consistent with $r+s+t=9$.

If nine points are given, we have linear equations to determine all the coefficients and $N(9, 0, 0)=1$.

If eight points are given, the equation of the surface can be thrown into the form $\alpha U+\beta V=0$, where U and V are any surfaces through the eight points. But the condition that a quadric should touch a line contains the coefficients in the second degree, and that it should touch a plane contains them in the third. We have therefore a quadratic in the one case and a cubic in the other, to determine the ratio $\alpha:\beta$, and $N(8, 1, 0)=2$, $N(8, 0, 1)=3$.

In like manner, if seven points are given, the equation of the quadric may be written $\alpha U+\beta V+\gamma W=0$, and, as before, we have $N(7, 2, 0)=4$, $N(7, 1, 1)=6$, $N(7, 0, 2)=9$.

So again, when six points are given, we have $N(6, 3, 0)=8$, $N(6, 2, 1)=12$, $N(6, 1, 2)=18$.

The first exception to the uniformity of the process occurs when the problem is to describe a quadric passing through six points and touching three planes. Now when a quadric reduces to a pair of planes, it is to be considered as touching

every plane. This is either evident geometrically, or may be seen by observing that the condition that a quadric should touch a plane vanishes identically when the quadric reduces to two planes. The ten pairs of planes then that can be described through the six points count among the twenty-seven apparent solutions of the problem, and there are but seventeen proper quadrics which can be described through the six points to touch the three planes. Hence $N(6, 0, 3) = 17$.

2. A pair of planes does not touch a right line unless the right line pass through the intersection of the planes. Hence, when five points and more lines than one are given, none of the pairs of planes count among the solutions, and $N(5, 4, 0) = 16$, $N(5, 3, 1) = 24$, $N(5, 2, 2) = 36$. When, however, we are given five points, one line, and three planes; if we take a plane through three points, and another through the remaining two points and through the point where the first plane meets the line, we have a pair of planes which fulfils the conditions of the problem. And there are evidently ten such pairs of planes which can be drawn to fulfil the conditions of the problem. But we shall show that these systems of planes count double among the solutions of the problem. First, if the coefficient of the highest power of α vanish, in any equation $\phi(\alpha, \beta, \gamma, \delta, \varepsilon) = 0$, then the point $\beta\gamma\delta\varepsilon$ satisfies that equation; and the coefficient of the next highest power of α, which is linear in $\beta, \gamma, \delta, \varepsilon$, is what we shall call the tangential quantic at this point. Now, if the point $\beta\gamma\delta\varepsilon$ be on four quantics, it will count double among their intersections if the four tangential quantics be connected by an identical linear relation. For then the equations can be so combined that the coefficients of the two highest powers of α shall disappear from one of them, or that the point $\beta\gamma\delta\varepsilon$ shall be a double point in that one, and therefore count doubly among the intersections. Now if we form the condition that a plane shall touch $\alpha U + \beta V + \gamma W + \delta Y + \varepsilon Z$, where U, V, W, Y, Z are five quadrics passing through five given points, it will be of the third degree in $\alpha\beta\gamma\delta\varepsilon$, and the coefficient of α^3 will vanish if U represent two planes. The coefficient of α^2 is $(\beta V_1 + \gamma W_1 + \delta Y_1 + \varepsilon Z_1)$, where V_1, W_1, &c. are the results of substituting in V, W, &c. the coordinates of the intersection of the touching plane with the intersection of the two planes U. Similarly, if we form the condition that a line shall touch $\alpha U + \beta V +$ &c., the coefficient of α^2 vanishes when U reduces to two planes, and the line passes through their intersection; and in that case the coefficient of the next highest power of

α is $\beta V_4 + \gamma W_4 + \delta Y_4 + \varepsilon Z_4$, where V_4 is the result of substituting in V the co-ordinates of the point where the line meets the intersection of the two planes U. But if we substitute in any quadric U the co-ordinates of four points on a right line, the results U_1, U_2, U_3, U_4 are connected by an identical relation in which the coefficients of the quadric do not enter. Thus, then, the conditions that the quadric should touch three planes, and a line, when U represents two planes, have the point $\beta\gamma\delta\varepsilon$ common, and the tangential quantics at that point connected by an identical relation. Therefore $\beta\gamma\delta\varepsilon$ counts double; and we have $N(5, 1, 3) = 54 - 2 \times 10 = 34$.

3. Let us next consider how many quadrics can be described through five points to touch four planes. If $\alpha U + \beta V + \gamma W + \delta Y + \varepsilon Z$ represent a quadric through the five points, we have four relations of the third degree connecting α, β, γ, δ, ε, apparently having eighty-one systems of common values. Now, let U be the product of the two planes (123), (145), $V = (123)\,(245)$, $W = (145)\,(234)$; then, in the first place, it is evident that the coefficients of α^3, $\alpha^2\beta$, $\alpha\beta^2$, β^3 must vanish in every one of the conditions, since, if $\gamma = 0$, $\delta = 0$, $\varepsilon = 0$, the quadric represents two planes. All four quantics then have common what in the paper "on some points in the Theory of Elimination" (vol. VII., pp. 327–337) we have called a *curve*, viz. *the right line* $\gamma = 0$, $\delta = 0$, $\varepsilon = 0$. This may be called the right line (123) (45). Further, it is evident that there are ten such lines common to the four quadrics, (123) (45), (124) (35), &c. It appears, in like manner, that the right line (145) (23) is denoted by $\beta = 0$, $\delta = 0$, $\varepsilon = 0$. And this line and (123) (45) have common the point $\beta = 0$, $\gamma = 0$, $\delta = 0$, $\varepsilon = 0$. It is seen thus that of the ten lines each line (123) (45) is intersected by three others, viz. (145) (23), (245)(13), (345)(12). There are in all fifteen points of intersection. The diminution then effected by such a system of lines is, according to the preceding paper, $10(l + m + n + r - 3) - 2(15)$. But since $l = m = n = r = 3$, the diminution is sixty, and we thus find $N(5, 0, 4) = 21$.

4. When we consider a system of quadrics through four points, it appears, as before, that $N(4, 5, 0) = 32$, $N(4, 4, 1) = 48$, $N(4, 3, 2) = 72$. The first case requiring special consideration is when it is required to describe a quadric through four points to touch two lines and three planes, a problem, the number of solutions of which evidently cannot exceed 108. But among these 108 will be reckoned two systems of two planes.

First, take the plane through three points, and the plane through the fourth, and through the two points where the plane meets the two lines. We have thus four systems of two planes which count among the solutions of the problem. But, secondly, there can be drawn two transversals to meet the four lines, 12, 34, and the two given lines. The planes passing through either transversal and the lines 12, 34, also satisfy the conditions of the problem. And since we might have divided the points 13, 24, or 14, 23, there are six systems of two planes whose intersection meets both given lines, and which pass each through two of the given points. It remains to consider how often these systems of planes count in the solutions of the problem. Now it appears, exactly as in Art. 2, that if we form the conditions that three planes and two lines should touch a system of quadrics one of which is a pair of planes whose intersection meets both lines, the tangential quantics corresponding to this common point are connected by two identical relations, and the point of intersection counts for four.* I conclude, therefore, that $N(4, 2, 3) = 108 - 4 \times 10 = 68$.

5. Let us now consider the problem to determine the number $N(4, 1, 4)$.

Any quadric through the four points formed by the tetrahedron $xyzw$ is

$$lyz + mzx + nxy + pxw + qyw + rzw = 0.$$

The condition that this surface should touch a plane is

$$rql\alpha^2 + pmr\beta^2 + nqp\gamma^2 + lmn\delta^2 + (p\beta\gamma + l\alpha\delta)(lp - mq - nr)$$
$$+ (q\gamma\alpha + m\beta\delta)(mq - nr - lp) + (r\alpha\beta + n\gamma\delta)(nr - lp - mq) = 0.$$

This will be satisfied always by the suppositions

$$(l = 0,\ q = 0,\ r = 0),\quad (p = 0,\ m = 0,\ r = 0),$$
$$(n = 0,\ q = 0,\ p = 0),\quad (l = 0,\ m = 0,\ n = 0),$$

no matter what $\alpha, \beta, \gamma, \delta$ be. In fact, in any of these cases the surface resolves itself into two planes, one of them being one of the planes of reference.

Again, it will also be satisfied by any of the suppositions $(l=0, p=0, mq=nr)$, $(m=0, q=0, nr=lp)$, $(n=0, r=0, lp=mq)$. In any of these cases the surface also resolves itself into two planes, passing through a pair of opposite edges of the tetrahedron of reference. When, therefore, we are given four planes touched by the surface, we have four equations of

the form $rqlx^2 + \&c. = 0$. And considering l, m, n, p, q, r as the variables to be determined, the equations being each of the third degree, their combination would form a system of the degree 81. But we have seen that the four quantics have common four *surfaces* of the first degree, viz. $l = 0$, $q = 0$, $r = 0$, &c., and three of the second, viz. $l = 0$, $p = 0$, $mq = nr$, &c. It is to be observed that any of the surfaces of the first degree have a right line common with any of the second; as, for instance, the two just written have common the line $l = 0$, $p = 0$, $q = 0$, $r = 0$. The system of surfaces then is one whose order is $4 \times 1 + 3 \times 2 = 10$, and whose weight is $4 \times 3 + 3 \times 8 + 2 \times 12 = 60$. But if four quantics in five variables, each of the third degree, have common a surface whose order and weight are 10 and 60, their complementary curve of intersection will be of the order $81 - (10 \times 12) + 60 = 21$. If now we are given a fifth point on the quadric, this gives another linear relation between l, m, &c., and the result already arrived at is confirmed; namely, that the problem to describe a quadric through five points to touch four planes admits of twenty-one solutions. If the quadric is to touch a line, we shall have a relation of the second degree in l, m, &c., which has therefore forty-two points common with the complementary curve. Hence $N(4, 1, 4) = 42$.

Knowing that $N(4, 0, 5) = 21$, it follows, reciprocally, that $N(5, 0, 4)$ is 21, but it would be desirable to obtain this result directly. The conditions that five planes should touch, form a system of five quantics each of the third degree, having common a system of surfaces whose order, as we have seen, is 10, whose weight is 60, and if the rank of the system be γ, the number of points of intersection would be

$$243 - 10 \times 90 + 60 \times 15 - \gamma.$$

Thus it is seen that the results already arrived at would be verified if it can be proved independently that γ is $= 222$: that is to say, that this is the rank of the system formed by the four planes, $l = 0$, $q = 0$, $r = 0$, &c., and the three quadrics $l = 0$, $p = 0$, $mq = nr$, &c. But this I have not succeeded in showing satisfactorily.

6. If a quadric reduces to two coincident planes, such a system touches every plane and every line. Consequently, when only three points on a quadric are given, and it is required to touch s lines and t planes $(s + t = 6)$, the double plane through the three points counts among the solutions of

the problem. Thus, to find the number $N(3, 6, 0)$, let us suppose that the quadric passes through the points xzw, xyz, xyw; or, in other words, that the coefficients b, c, d are wanting. Then in the condition that the quadric should touch a line, the coefficient of a^2 is wanting, and the coefficient of a is $\frac{l}{(\beta\gamma')} + \frac{q}{(\delta\beta')} + \frac{r}{(\gamma\delta')}$.

Thus, when there are six such conditions, the tangential quantics are connected by three identical relations, and the point $l=0$, $m=0$, $n=0$, $p=0$, $q=0$, $r=0$ counts for eight among the intersections. We have then $N(3, 6, 0)=56$.

Let us next examine the number $N(3, 5, 1)$.

We have then five conditions that the quadrics should touch right lines, three in which it may be supposed that the coefficients of a are l, q, r, and two in which a does not appear. In the condition that the quadric should touch a plane, we have neither a^3 nor a^2; therefore the point $lmnpqr$ is a double point in this condition; but since the coefficient of a contains only l, q, r, it appears to me that this operates as a triple point, and that the reduction is 12.* Since, in general, a line cannot be drawn to intersect five lines, there appears to be no further reduction, and $N(3, 5, 1)=84$.

In like manner we find $N(3, 4, 2)=126$.

In examining $N(3, 3, 3)$, we find that the point $lmnpqr$ counts for twenty-seven.† But there are besides six systems of planes whose intersection meets all these lines, and which pass one through two points and the other through the remaining one. These planes count each for eight; whence $N(3, 3, 3)=141$.

The other numbers $N(3, 2, 4)$, $N(3, 1, 5)$, $N(3, 0, 6)$ are the reciprocals of numbers already obtained.

7. To find $N(2, 7, 0)$ we suppose the coefficients c and d to vanish in the equation of a quadric. The condition then that a line should touch vanishes if the quadric should reduce to $(\alpha x+\beta y)^2$; that is to say, it contains the conic

$$l=0,\ m=0,\ p=0,\ q=0,\ r=0,\ n^2=ab.$$

But the quantics will touch along the curve, and therefore the formulæ of the preceding paper are not applicable to this case. The same case arises in the treatment of the problem

* It appears from M. Chasles's results that this should be 16, which alters $N(3, 5, 1)$ and $N(3, 4, 2)$ to 80 and 112.

† In like manner this number should be 64 which alters $N(3, 3, 3)$ to 104.

to find how many conics can be described through a point to touch four lines. I have obtained formulæ for quantics touching along curves, only in some of the simplest cases. I therefore leave undiscussed the determination of $N(r, s, t)$, when neither r, nor t, exceed 2.*

NOTE ON THE COMPOSITION OF INFINITESIMAL ROTATIONS.

By Prof. Cayley.

THE following is a solution of a question proposed by me in the last Smith's Prize Examination:

"Show that infinitesimal rotations impressed upon a solid body may be compounded together according to the rules for the composition of forces."

Def. The "six coordinates" of a line passing through the point (x_0, y_0, z_0), and inclined at angles (α, β, γ), to the axes, are

$$\begin{aligned} a &= \cos\alpha, & f &= y_0 \cos\gamma - z_0 \cos\beta, \\ b &= \cos\beta, & g &= z_0 \cos\alpha - x_0 \cos\gamma, \\ c &= \cos\gamma, & h &= x_0 \cos\beta - y_0 \cos\alpha. \end{aligned}$$

I use, throughout, the term rotation to denote an infinitesimal rotation; this being so,

Lemma 1. A rotation ω round the line (a, b, c, f, g, h), produces in the point (x, y, z), rigidly connected with the line, the displacements

$$\begin{aligned} \delta x &= \omega\,(\quad . \quad cy - bz + f), \\ \delta y &= \omega\,(-cx \quad . \quad + az + g), \\ \delta z &= \omega\,(\ bx - ay \quad . \quad + h). \end{aligned}$$

Lemma 2. Considering in a solid body the point (x, y, z), situate in the line (a, b, c, f, g, h), then for any infinitesimal

* This paper was written some months ago, and was intended to appear together with the paper above referred to, which was printed in the last number of the Journal. Since then M. Chasles has published (as yet, however, without demonstration) determinations of all the numbers $N(r, s, t)$. I have not suppressed this paper, since it is likely that the course followed by M. Chasles is quite different from mine.

motion of the solid body, the displacement of the point in the direction of the line is

$$= al + bm + cn + fp + gq + hr,$$

where l, m, n, p, q, r are constants depending on the infinitesimal motion of the solid body.

Hence, *first*, for a system of rotations

$$\begin{array}{lll} \omega_1 & \text{about the line} & (a_1, b_1, c_1, f_1, g_1, h_1), \\ \omega_2 & \quad\text{"} \quad\quad \text{"} & (a_2, b_2, c_2, f_2, g_2, h_2), \\ \text{\&c.} & & \end{array}$$

the displacements of the point (x, y, z), are

$$\begin{aligned} \delta x &= \quad . \quad\quad\;\; y\Sigma c\omega - z\Sigma b\omega + \Sigma f\omega, \\ \delta y &= -x\Sigma c\omega \quad . \quad + z\Sigma a\omega + \Sigma g\omega, \\ \delta z &= \;\; x\Sigma b\omega + y\Sigma a\omega \quad . \quad + \Sigma h\omega; \end{aligned}$$

and when the rotations are in equilibrium, the displacements $(\delta x, \delta y, \delta z)$ of any point (x, y, z) whatever must each of them vanish; that is, we must have

$$\Sigma\omega a = 0, \;\; \Sigma\omega b = 0, \;\; \Sigma\omega c = 0, \;\; \Sigma\omega f = 0, \;\; \Sigma\omega g = 0, \;\; \Sigma\omega h = 0,$$

which are therefore the conditions for the equilibrium of the rotations ω_1, ω_2, &c.

Secondly, for a system of forces

$$\begin{array}{lll} P_1 & \text{along the line} & (a_1, b_1, c_1, f_1, g_1, h_1), \\ P_2 & \quad\text{"} \quad\quad \text{"} & (a_2, b_2, c_2, f_2, g_2, h_2), \\ \text{\&c.} & & \end{array}$$

the condition of equilibrium as given by the principle of virtual velocities is

$$\Sigma P(al + bm + cn + fp + gq + hr) = 0;$$

or, what is the same thing, we must have

$$\Sigma Pa = 0, \;\; \Sigma Pb = 0, \;\; \Sigma Pc = 0, \;\; \Sigma Pf = 0, \;\; \Sigma Pg = 0, \;\; \Sigma Ph = 0,$$

which are therefore the conditions for the equilibrium of the forces P_1 P_2, &c.

Comparing the two results we see that the conditions for the equilibrium of the rotations ω_1, ω_2, &c. are the same as those for the equilibrium of the forces P_1, P_2, &c.; and since, for rotations and forces respectively, we pass at once from the theory of equilibrium to that of composition; the rules of composition are the same in each case.

Demonstration of Lemma 1.

Assuming for a moment that the axis of rotation passes through the origin, then for the point P, coordinates (x, y, z), the square of the perpendicular distance from the axis is

$$\begin{aligned} =(\ \ \cdot\ \ &-y\cos\gamma+z\cos\beta)^2\\ +(\ x\cos\gamma\ \ \cdot\ \ &-z\cos\alpha)^2\\ +(-x\cos\beta+y\cos\alpha\ \ \cdot\ \ &)^2, \end{aligned}$$

and the expressions which enter into this formula denote as follows; viz. if through the point P, at right angles to the plane through P and the axis of rotation, we draw a line PQ, = perpendicular distance of P from the axis of rotation, then the coordinates of Q referred to P as origin are

$$\begin{array}{c} \cdot\ \ -y\cos\gamma+z\cos\beta,\\ x\cos\gamma\ \ \cdot\ \ -z\cos\alpha,\\ -x\cos\beta+y\cos\alpha\ \ \cdot\ \ , \end{array}$$

respectively. Hence the foregoing quantities each multiplied by ω are the displacements of the point P in the directions of the axes, produced by the rotation ω. Suppose that the axis of rotation (instead of passing through the origin) passes through the point (x_0, y_0, z_0); the only difference is that we must in the formulæ write $(x-x_0, y-y_0, z-z_0)$ in place of (x, y, z): and attending to the significations of the six coordinates (a, b, c, f, g, h) it thus appears that the displacements produced by the rotation are equal to ω into the expressions

$$\begin{array}{c} \cdot\ \ -cy+bz+f,\\ cx\ \ \cdot\ \ -az+g,\\ -bx+ay\ \ \cdot\ \ +h, \end{array}$$

respectively.

Demonstration of Lemma 2.

For any infinitesimal motion whatever of a solid body, the displacements of the point (x, y, z) in the directions of the axes are

$$\begin{aligned} \delta x&=l\ \ \cdot\ \ -ry+qz,\\ \delta y&=m+rx\ \ \cdot\ \ -pz,\\ \delta z&=n-qx+py\ \ \cdot\ \ , \end{aligned}$$

and hence the displacement in the direction of the line (α, β, γ) is

$$\delta x\cos\alpha+\delta y\cos\beta+\delta z\cos\gamma,$$

which, attending to the signification of the six coordinates (a, b, c, f, g, h) is

$$= al + bm + cn + fp + gq + hr,$$

which is the required expression.

It is proper to remark that the last-mentioned expressions of $(\delta x, \delta y, \delta z)$ are in fact the displacements produced by a translation and a rotation. If we *assume* that every infinitesimal motion of a solid body can be resolved into a translation and a rotation, then, since a translation can be produced by two rotations, every infinitesimal motion of a solid body can be resolved into rotations alone, and the foregoing expressions for the displacements produced by a rotation, combining any number of them and writing $(\Sigma\omega a, \Sigma\omega b, \Sigma\omega c, \Sigma\omega f, \Sigma\omega g, \Sigma\omega h) = (-p, -q, -r, l, m, n)$ respectively, lead to the expressions for the displacements δx, δy, δz produced by the infinitesimal motion of the solid body.

ON A PROPERTY OF THE DIRECTOR SPHERES OF A SYSTEM OF QUADRICS TOUCHING A COMMON SYSTEM OF PLANES.

By R. Townsend, M.A.

The following analogue in the Theory of Quadrics to the property of Conics given by Mr. Ferrers in the *Messenger of Mathematics*, Vol. I., p. 159, may be interesting to the reader.

If a system of quadrics touch a common system of eight, seven, or six planes, their director spheres (that is, the spheres loci of the intersections of their rectangular triads of tangent planes) have a common radical plane, axis, or centre.

To prove the first property: denoting by A any one of the eight planes, by P either of the two points on A common to the two director spheres of any two of the quadrics, by L the line through P perpendicular to A, and by X and X', Y and Y', Z and Z', the three pairs of planes through L tangent to the same two and to any third of the quadrics; then since, by a known property of quadrics having a common circumscribing developable (see Salmon's *Geometry of Three Dimensions*, second edition, Art. 126), the three pairs of planes X and X', Y and Y', Z and Z' are in involution, and

since, by hypothesis (the point P being on the director spheres of the first two quadrics), the two pairs X and X', Y and Y' are rectangular, therefore the third pair Z and Z' is also rectangular, and therefore the point P is on the director sphere of the third quadric also; and the same being true of each of the seven corresponding points Q, R, S, &c. on the seven remaining planes B, C, D, &c., therefore, &c.

N.B. The known property (see Salmon's *Geometry of Three Dimensions*, second edition, Art. 128), that the locus of the centres of the system of quadrics which touch the same eight planes is a right line, follows of course at once from that just established.

Cor. 1°. For the particular quadric of the system whose centre is at infinity, that is, for the paraboloid which touches the eight planes, the director sphere opens out into a plane, which is consequently that of the eight points, P, Q, R, S, &c. Hence—

If a system of quadrics touch the same eight planes, the common radical plane of their director spheres is the director plane of the paraboloid which touches the planes.

Cor. 2°. When one of the eight planes is at infinity, and when consequently every quadric that touches them all is necessarily a paraboloid, the director spheres are all planes, and the several points P, Q, R, S, &c., are consequently all collinear; Hence—

If a system of paraboloids touch the same seven planes, their director planes have a common line of intersection.

Cor. 3°. When three of the eight planes have a common line of intersection, every quadric that touches them all being then necessarily a ruled quadric passing through the line; Hence—

If a system of ruled quadrics passing through a common line touch five common planes, their director spheres have a common radical plane.

Cor. 4°. When, of the eight planes, three have a common line of intersection, and three more another common line of intersection, every quadric that touches them all being then necessarily a ruled quadric passing through the two lines; Hence—

If a system of ruled quadrics pass through two common lines and touch two common planes, their director spheres have a common circle of intersection with that of the ruled quadric

passing through the two lines and through the intersection of the two planes.

COR. 5°. When the eight planes pass in pairs through the four sides of any skew quadrilateral, every ruled surface passing through the four sides then necessarily touching them all, and the two diagonals being infinitely slender surfaces also touching them all; Hence—

The director sphere of every ruled quadric passing through the four sides of any skew quadrilateral passes through the circle of intersection, real or imaginary, of the two spheres of which the two diagonals are diameters.

N.B. The known property, that the centre of every ruled quadric passing through the four sides of any skew quadrilateral lies on the right line connecting the middle points of the two diagonals, follows of course immediately from this last.

COR. 6°. When the eight planes pass in pairs through any four lines through which the same ruled quadric can pass, the quadric so passing then necessarily touching them all; Hence—

When eight planes pass in pairs through any four generators of the same ruled quadric, the director spheres of all quadrics touching them all have a common circle of intersection, real or imaginary, with that of the original quadric.

To prove the second property. Since, by the first property (see Cor. 1° above), the radical plane of the director spheres of any two quadrics touching the seven planes is the director plane of the paraboloid touching them, whose centre is the point at infinity in the direction collinear with the centres of the two spheres; and since, by the same again (see Cor. 2° above), the director planes of all paraboloids touching the same seven planes have a common line of intersection; therefore, &c.

N.B. The known property (see Salmon's *Geometry of Three Dimensions*, second edition, Art. 131), that the locus of the centres of all quadrics which touch the same seven planes is a plane, follows of course at once from the above.

COR. 1°. When one of the seven planes is at infinity, and when consequently every quadric that touches them all is necessarily a paraboloid, the director spheres being then all planes, and every system of spheres passing through two common points becoming when their plane of centres is a

infinity a system of planes passing through one common point; Hence—

If a system of paraboloids touch the same six planes, their director planes have a common point of intersection.

Cor. 2°. When three of the seven planes have a common line of intersection, every quadric (not infinitely slender) which touches them all being then necessarily a ruled quadric passing through the line; Hence—

If a system of ruled quadrics passing through a common line touch four common planes, their director spheres have a common radical axis.

Cor. 3°. In the same case, the four lines, connecting the four intersections of the common line with the remaining four planes to the opposite vertices of the tetrahedron determined by those planes, being infinitely slender surfaces touching the seven planes; Hence—

When four points on the four faces of a tetrahedron are collinear, the four spheres, of which the four lines connecting them with the opposite vertices are diameters, have a common radical axis.

N.B. This latter property is due to Mr. Clifford, who proposed it for solution in the Educational Times of August, 1865; a very elegant analytical demonstration of it by Professor Cayley appeared shortly after in the same periodical.

Cor. 4°. When, of the seven planes, three have a common line of intersection, and three more another common line of intersection, then every quadric (not infinitely slender) which touches them all being necessarily a ruled quadric passing through the two lines, and the line connecting the two intersections of the latter with the remaining plane being an infinitely slender surface touching them all; Hence—

If a system of ruled quadrics pass through two common lines and touch a common plane, their director spheres have a common chord of intersection with the sphere of which the interval between the intersections of the lines with the plane is a diameter.

To prove the third property. Since, by property 1, Cor. 1°, the radical plane of the director spheres of any two quadrics touching the six planes, is the director plane of a certain paraboloid touching them, whose centre is the point at infinity in the direction collinear with the centres of the two spheres; and since, by property 2, Cor. 1°, the director

planes of all paraboloids touching the same six planes have a common point of intersection; therefore &c.

COR. 1°. Since the ruled quadric determined by any triad of intersections of the six planes taken in pairs touches the six planes, and since fifteen such triads exist for different groupings of the six planes in pairs; Hence—

The director sph res of all quadrics touching six common planes, have a common radical centre with those of the fifteen ruled quadrics, determined by the fifteen different triads of intersections of the planes taken in pairs.

COR. 2°. When three of the six planes have a common line of intersection, every quadric (not infinitely slender) which touches them all being then necessarily a ruled quadric passing through the line; and every connector of a point on the line with the intersection of the remaining three planes being an infinitely slender surface touching them all; Hence—

If a system of ruled quadrics pass through a common line and touch three common planes, their director spheres have a common radical centre with the system of spheres of which the several lines connecting the intersection of the planes with the several points of the line are diameters.

COR. 3°. When, of the six planes, three pass through a common line, and the remaining three through another common line, every quadric (not infinitely slender) which touches them all being then necessarily a ruled quadric passing through the two lines; and every line terminated by both lines being an infinitely slender surface touching them all; Hence—

If a system of ruled quadrics pass through two common lines, their director spheres have a common radical centre with the system of spheres which intersect the two lines at diametrically opposite points.

N.B. This last property is evident *à priori* from the obvious consideration that the centres of both systems of spheres lie in the plane parallel to the two lines which bisects at right angles the common perpendicular to them.

Trinity College, Dublin,
February 15*th*, 1866.

NOTE ON THE NINE-POINT CIRCLE.

By JOHN GRIFFITHS, M.A.

LET O be the centre of the circumscribing circle of any triangle ABC; P the point of intersection of its perpendiculars; $l, m, n,$ the middle points of the sides BC, CA, AB. Join O to l, m, n, and produce Ol, Om, On to A_1, B_1, C_1, so that $OA_1 = 2Al$, $OB_1 = 2Om$, $OC_1 = 2An$.

Then the nine-point circle of the triangle ABC is the auxiliary circle of each of the four conics which touch the sides of the triangles ABC, PBC, PCA, PAB, respectively, and whose foci coincide with the following pairs of points:

$$1^{\circ}.\ O, P, \quad 2^{\circ}.\ A, A_1, \quad 3^{\circ}.\ B, B_1, \quad 4^{\circ}.\ C, C_1.$$

These theorems can be immediately deduced from the following more general one:

The feet of the perpendiculars let fall from the two points whose trilinear coordinates are λ, μ, ν, and $\frac{1}{\lambda}, \frac{1}{\mu}, \frac{1}{\nu}$ upon the sides of the triangle of reference lie on the circumference of the same circle, viz., the auxiliary circle of the conic which touches the sides of this triangle, and whose foci are the above two points.

The trilinear equation of this six-point circle is

$$\begin{aligned}
&\lambda(\mu+\nu\cos A)(\mu\cos A+\nu)\alpha^2\\
&+\mu(\nu+\lambda\cos B)(\nu\cos B+\lambda)\beta^2\\
&+\nu(\lambda+\mu\cos C)(\lambda\cos C+\mu)\gamma^2\\
&-\{\mu(\lambda\cos C+\mu)(\lambda+\nu\cos B)+\nu(\lambda+\mu\cos C)(\lambda\cos B+\nu)\}\beta\gamma\\
&-\{\nu(\mu\cos A+\nu)(\mu+\lambda\cos C)+\lambda(\mu+\nu\cos A)(\mu\cos C+\lambda)\}\gamma\alpha\\
&-\{\lambda(\nu\cos B+\lambda)(\nu+\mu\cos A)+\mu(\nu+\lambda\cos B)(\nu\cos A+\mu)\}\alpha\beta=0.
\end{aligned}$$

If $\frac{\lambda}{\cos A}=\frac{\mu}{\cos B}=\frac{\nu}{\cos C}$, this equation represents the nine-point circle; and if $\lambda=\mu=\nu$, we have the equation of the inscribed circle, and so on.

Kidwelly, South Wales, ***March 31st,*** **1865.**

ANALYTICAL METRICS.

By W. K. Clifford.

(Continued from Vol. vii., p. 67.)

IV. Determination of the Absolute.

12. As the title of this section might be productive of mixed feelings, I will say at once that "to determine the Absolute" means "to find the form of ϕ." It is not pretended that the method here used is simpler than that generally given; but only that it is more suggestive and of wider application.

Trilinears.

The trilinear coordinates of a point are three quantities proportional to the perpendicular distances of the point from three given lines, $X=0$, $Y=0$, $Z=0$. We may therefore write, if x, y, z are these coordinates,

$$x,\, y,\, z = \frac{X}{\sqrt{\phi X}},\ \frac{Y}{\sqrt{\phi Y}},\ \frac{Z}{\sqrt{\phi Z}},$$

and so

$$\begin{aligned}\phi\,(lx+my+nz) &= l^2\phi x + m^2\phi y + n^2\phi z + 2mn\psi yz + 2nl\psi zx + 2lm\psi xy \\ &= l^2 + m^2 + n^2 + 2mn\cos(YZ) + 2nl\cos(ZX) + 2lm\cos(XY).\end{aligned}$$

It only remains to determine *which* cosine is meant in each case. To do this we make the convention that x, y, z shall be all positive for a point inside the triangle. We can then see geometrically, that a line perpendicular to BC (or X) through B (or ZX), is represented by

$$z + x\cos B = 0,$$

therefore

$$\psi\,(x,\, z + x\cos B) \equiv \psi\,(x, z) + \cos B = 0, \text{ or } \cos(XZ) = -\cos B.$$

Hence, by symmetry, we have,

$$\phi\,(lx+my+nz) = l^2 + m^2 + n^2 - 2mn\cos A - 2nl\cos B - 2lm\cos C,$$

where A, B, C are the internal angles of the triangle.

One may now safely see, *a priori*, that the angle between two straight lines *ought* to be the angle through which one

of them must be turned in order that its positive side may coincide with the positive side of the other. And by aid of this definition the result we have just found may be extended from Trilinear to Multilinear systems. We shall always have

$$\phi\,(lx+my+nz+\ldots)\equiv l^2+m^2+\ldots-2lm\cos(xy)-\ldots$$

Areals.

The Areal coordinates of a point are quantities proportional to the triangles it determines with the sides of a certain fixed triangle. Let α, β, γ be the sides of this triangle; then we have

$$x,\,y,\,z=\frac{\alpha X}{\sqrt{\phi X}},\ \frac{\beta Y}{\sqrt{\phi Y}},\ \frac{\gamma Z}{\sqrt{\phi Z}},$$

and so

$$\phi\,(lx+my+nz)$$
$$=\alpha^2l^2+\beta^2m^2+\gamma^2n^2-2\beta\gamma mn\cos A-2\gamma\alpha nl\cos B-2\gamma\alpha lm\cos C,$$

the signs being determined as before.

Interpretation of Constants.

13. Let P_1, P_2, P_3 be the three perpendiculars of the triangle of reference, and ϖ_1, ϖ_2, ϖ_3 the perpendiculars from the angular points on the line (lmn). Then by the formula in Art. 7, we have, in Trilinears,

$$\varpi_1:\varpi_2:\varpi_3=lP_1:mP_2:nP_3,$$

since, in this system, the coordinates of the angular points are as

$$P_1,\,0,\,0;\quad 0,\,P_2,\,0;\quad 0,\,0,\,P_3.$$

Consequently

$$l:m:n=\frac{\varpi_1}{P_1}:\frac{\varpi_2}{P_2}:\frac{\varpi_3}{P_3}.$$

In the Areal system, the coordinates of the angular points are as 1, 0, 0; 0, 1, 0; 0, 0, 1; so that in this case we have

$$l:m:n=\varpi_1:\varpi_2:\varpi_3.$$

It is of the greatest importance to notice that there are *two different* systems of Tangential Coordinates, corresponding to Trilinears and Areals respectively. In the former, the coordinates of a line are proportional to $\frac{\varpi_1}{P_1}$, $\frac{\varpi_2}{P_2}$, $\frac{\varpi_3}{P_3}$; in the

* This rule determines the signs in several of the formulæ of III.

latter, they are proportional to $\varpi_1, \varpi_2, \varpi_3$. If we keep this distinction in mind, we may say generally that the *coefficients* in the *equation* of a point are proportional to the *coordinates* of the point; and that the *coefficients* in the *equation* of a line are proportional to the *coordinates* of the line. And then we may get rid of coordinates altogether, and consider nothing but equations.

Line at Infinity.

14. Form the reciprocal of ϕ by the ordinary method; it is, in Trilinears,

$$(x \sin A + y \sin B + z \sin C)^2,$$

and in Areals,

$$\beta^2\gamma^2 \sin^2 A\,(x + y + z)^2.$$

It must be remembered that we are not here finding the *equation* of the line at infinity, but the value of the function $A\infty^2$ of the point (x, y, z). In the case of Trilinears, if we suppose x, y, z to be actually *equal* to the perpendiculars from a point on the sides, then the quantity

$$x \sin A + y \sin B + z \sin C$$

is four times the projector of the triangle of reference.

Quadrilaterals.

15. Consider four straight lines $A, B, C, D = 0$. Since the equation of *any* line can be expressed in terms of the equations of three others, there must be an identical relation

$$lA + mB + nC + sD \equiv 0 \text{ .... } (1),$$

and because this is independent of the values of x, y, z, we must have

$$lA_x + mB_x + nC_x + sD_x = 0,$$
$$lA_y + mB_y + nC_y + sD_y = 0,$$
$$lA_z + mB_z + nC_z + sD_z = 0,$$

where A_x denotes the coefficient of x in A, &c. Eliminating therefore l, m, n, s from these four equations, we have

$$\begin{vmatrix} A, & B, & C, & D \\ A_x, & B_x, & C_x, & D_x \\ A_y, & B_y, & C_y, & D_y \\ A_z, & B_z, & C_z, & D_z \end{vmatrix} \equiv 0.$$

But, by definition of a Jacobian, this is equivalent to

$$A.J(BCD)-B.J(CDA)+C.J(DAB)-D.J(ABC)\equiv 0,$$

and this therefore is the identical relation between four given lines.

Now it is often convenient to use a set of four coordinates, x, y, z, w, connected by the identical relation

$$x+y+z+w=0;$$

we have then only to put

$$x, y, z, w=A.J(BCD),\ B.J(CAD),\ C.J(ABD),\ D.J(CBA),$$

and we shall have such a system. We then find that

$$\phi\,(lx+my+nz+sw)$$
$$=l^2\phi A.J(BCD)^2+\ldots+2lm.\psi AB.J(BCD).J(CAD)+\ldots$$

Now let α, β, γ, δ be the projectors of the triangles BCD, CAD, ABD, CBA; and divide the result just found by $\phi A.\phi B.\phi C.\phi D$; then we may write

$$\phi(lx+my+nz+sw)=\alpha^2l^2+\beta^2m^2+\gamma^2n^2+\delta^2s^2-2\alpha\beta lm\cos(xy)-\ldots$$

We have here in fact made x proportional to the perpendicular from a point on A, multiplied by the projector of BCD. And this might be taken as the definition of the system of coordinates.

If for D we write ∞, the identity becomes

$$\infty.J(ABC)\equiv A.J(BC\infty)+B.J(CA\infty)+C.J(AB\infty).$$

Hence we find, in Trilinears, for instance,

$$\infty.\frac{J(ABC)}{\sqrt{\phi A.\phi B.\phi C}}\equiv x\sin A+y\sin B+z\sin C\equiv P(ABC),$$

which agrees with the result before obtained, since $\dfrac{J(ABC)}{\sqrt{\phi A.\phi B.\phi C}}$ is really the *ratio* of the projector of ABC to the projector of the triangle of reference.

Equations.

16. The *condition* that a point A shall lie on a line L is $AL=0$. Now if we consider the coefficients of the point as coordinates, then $AL=0$ is the *equation* of the *line* L; if we consider the coefficients of the line as coordinates, $AL=0$ is the equation of the *point* A. It is not difficult

to see how this notion may be generalized. Consider the form

$$\xi x + \eta y + \zeta z,$$

which may represent either a point or a line, according as $(\zeta\eta\xi)$ or (xyz) are regarded as variable. I shall denote this form by an asterisk (*), and use it exclusively to represent *equations*. For instance, the equation $J(AB^*) = 0$ is the equation to the *point* or *line* AB, according as A and B are lines or points. I give one or two examples of equations found by this method.

α) *Locus of a point subtended by four given points in a given anharmonic ratio.*

We want a point P such that $\dfrac{\sin APB . \sin CPD}{\sin APC . \sin DPB} = k$. The locus is then immediately written down; viz., since

$$\sin APB = \frac{P\infty . J(ABP)}{\sqrt{\phi(AP) . \phi(BP)}},$$

it is $\qquad J(AB^*) . J(CD^*) = k J(AC^*) . J(DB^*),$

which is clearly a conic through the points $ABCD$.

β) *Envelop of a line cut by four given lines in a given anharmonic ratio.*

By the formula, Art. 7, the envelop is (in Tangential Coordinates),

$$J(LM^*) . J(NR^*) = k . J(LN^*) . J(RM^*),$$

a conic touching the four given lines.

γ) *Envelop of a line, the product of whose distances from two fixed points is constant.*

Ans. $\qquad (A^*)\,(B^*) = k^2 A\infty . B\infty . \phi\,(^*),$

a conic inscribed in the quadrilateral formed by joining A and B to the two points of ϕ.

δ) *Envelop of a line of constant length resting on two given lines.*

Ans. $\quad \phi\,(^*) . J(LM^*)^2 = k^2 . J(L\infty^*)^2 . J(M\infty^*).$

Let α be the point LM, β the point $L\infty$, γ the point $M\infty$; then if we remember that any point on the line $\alpha\beta$, *i.e.* at infinity, has its equation of the form $l\alpha + m\beta = 0$, this equation may be written

$$\alpha^2\,(a\beta^2 + 2b\beta\gamma + c\gamma^2) = k^2\beta^2\gamma^2,$$

or

$$\frac{a}{\gamma^2} + \frac{2b}{\beta\gamma} + \frac{c}{\beta^2} = \frac{k^2}{\alpha^2};$$

the envelop is therefore the tangential inverse, in respect of the triangle $\alpha\beta\gamma$, of a conic in respect of which α is the pole of $\beta\gamma$. It is obvious from the figure that each of the lines L, M, touches at two cusps, so that the curve is of the sixth order. The equation shews that the line at infinity $(\beta\gamma)$ is also a double tangent, and when L, M are at right angles, $b=0$, and the line at infinity touches at two cusps.

ε) *Envelop of a line cut by three given lines in given ratios.*

$$\textit{Ans.}\quad \lambda J(MN^*).J(L\infty^*)+\mu J(NL^*).J(M\infty^*)$$
$$+\nu J(LM^*).J(N\infty^*)=0,$$

a parabola touching the three given lines.

The utility of the method in questions of this sort is still more evident in the case of curves and surfaces of the second order.

(To be continued.)

ON $\Delta^m 0^n$ AND A FORM OF IT BY DR. HARGREAVE. SUMS OF POWERS OF NUMBERS, &c.

By George Scott, M.A., Trinity College, Dublin.

At the close of my paper in the 25th number of this *Journal*, I gave a rule for that kind of partition, in which repetitions of the same number are admissible. If we apply this rule to the expressions given in the first part of that paper for $\Delta^m 0^n$, and for the coefficients of the equation whose roots are the numbers 1, 2, 3 up to $n-1$, we may, *with proper restrictions*, write the results in both cases alike, thus:

$$\left.\begin{matrix}\dfrac{\Delta^{n-1}0^n}{\underline{n-1}}\\ P_1\end{matrix}\right\}=n(n-1)\left\{\frac{1}{2}\right\},$$

$$\left.\begin{matrix}\dfrac{\Delta^{n-2}0^n}{\underline{n-2}}\\ P_2\end{matrix}\right\}=n(n-1)(n-2)\left[\frac{1}{3}+\frac{n-3}{1.2}\left\{\frac{1}{2^2}\right\}\right],$$

$$\left.\begin{matrix}\dfrac{\Delta^{n-3}0^n}{\underline{|n-3}} \\ P_3\end{matrix}\right\} = n(n-1)(n-2)(n-3)\Bigg[\frac{1}{4} + \frac{n-4}{1.2}\left\{\frac{2}{2.3}\right\}$$

$$+ \frac{(n-4)(n-5)}{1.2.3}\left\{\frac{1}{2^3}\right\}\Bigg],$$

$$\left.\begin{matrix}\dfrac{\Delta^{n-4}0^n}{\underline{|n-4}} \\ .P_4\end{matrix}\right\} = n(n-1)\ldots(n-4)\Bigg[\frac{1}{5} + \frac{n-5}{1.2}\left\{\frac{2}{2.4} + \frac{1}{3^2}\right\}$$

$$+ \frac{(n-5)(n-6)}{1.2.3}\left\{\frac{3}{2^2.3}\right\} + \frac{(n-5)\ldots(n-7)}{1.2.3.4}\left\{\frac{1}{2^4}\right\}\Bigg],$$

$$\left.\begin{matrix}\dfrac{\Delta^{n-5}0^n}{\underline{|n-5}} \\ P_5\end{matrix}\right\} = n(n-1)\ldots(n-5)\Bigg[\frac{1}{6} + \frac{n-6}{1.2}\left\{\frac{2}{2.5} + \frac{2}{3.4}\right\}$$

$$+ \frac{(n-6)(n-7)}{1.2.3}\left\{\frac{3}{2^2.4} + \frac{3}{2.3^2}\right\} + \frac{(n-6)\ldots(n-7)}{1.2.3.4.5}\left\{\frac{1}{2^5}\right\}$$

$$+ \frac{(n-6)\ldots(n-8)}{1.2.3.4}\left\{\frac{4}{2^3.3}\right\}\Bigg],$$

$$\left.\begin{matrix}\dfrac{\Delta^{n-6}0^n}{\underline{|n-6}} \\ P_6\end{matrix}\right\} = n(n-1)\ldots(n-6)\Bigg[\frac{1}{7} + \frac{n-7}{1.2}\left\{\frac{2}{2.6} + \frac{2}{3.5} + \frac{1}{4^2}\right\}$$

$$+ \frac{(n-7)(n-8)}{1.2.3}\left\{\frac{3}{2^2.5} + \frac{1}{3^3}\right\} + \frac{(n-7)\ldots(n-9)}{1.2.3.4}\left\{\frac{4}{2^3.4} + \frac{6}{2^2.3^2}\right\}$$

$$+ \frac{(n-7)\ldots(n-10)}{1.2.3.4.5}\left\{\frac{5}{2^4.3}\right\} + \frac{(n-7)\ldots(n-11)}{1.2.3.4.5.6}\left\{\frac{1}{2^6}\right\}\Bigg],$$

$$\left.\begin{matrix}\dfrac{\Delta^{n-7}0^n}{\underline{|n-7}} \\ P_7\end{matrix}\right\} = n(n-1)\ldots(n-7)\Bigg[\Bigg\{\frac{1}{8} + \frac{n-8}{1.2}\left\{\frac{2}{2.7} + \frac{2}{3.6} + \frac{2}{4.5}\right\}$$

$$+ \frac{(n-8)(n-9)}{1.2.3}\left\{\frac{3}{2^2.6} + \frac{3.2}{2.3.5} + \frac{3}{2.4^2} + \frac{3}{3^2.4}\right\}$$

$$+ \frac{(n-8)(n-9)(n-10)}{1.2.3.4}\left\{\frac{4}{2^3.5} + \frac{4.3}{2^2.3.4} + \frac{4}{2.3^3}\right\}$$

$$+ \frac{(n-8)\ldots(n-11)}{1.2.3.4.5}\left\{\frac{5}{2^4.4} + \frac{5.2}{2^3.3^2}\right\}$$

$$+ \frac{(n-8)\ldots(n-12)}{1.2.3.4.5.6}\left\{\frac{6}{2^5.3}\right\} + \frac{(n-8)\ldots(n-13)}{1.2.3.4.5.6.7}\left\{\frac{1}{2^7}\right\}\Bigg],$$

$$
\left.\begin{array}{c}\dfrac{\Delta^{n-8}0^n}{\underline{\underline{n-8}}}\\ P_8\end{array}\right\} = n(n-1)\dots(n-8)\left[\frac{1}{9} + \frac{n-9}{1.2}\left\{\frac{2}{2.8} + \frac{2}{3.7} + \frac{2}{4.6} + \frac{1}{5^2}\right\}\right.
$$

$$
+ \frac{(n-9)(n-10)}{1.2.3}\left\{\frac{3}{2^2.7} + \frac{3.2}{2.3.6} + \frac{3.2}{2.4.5} + \frac{3}{3^2.5} + \frac{3}{3.4^2}\right\}
$$

$$
+ \frac{(n-9)\dots(n-11)}{1.2.3.4}\left\{\frac{4}{2^3.6} + \frac{4.3}{2^2.3.5} + \frac{3.2}{2^2.4^2} + \frac{4.3}{2.3^2.4} + \frac{1}{3^4}\right\}
$$

$$
+ \frac{(n-9)\dots(n-12)}{1.2.3.4.5}\left\{\frac{5}{2^4.5} + \frac{5.4}{2^3.3.4} + \frac{5.2}{2^2.3^3}\right\}
$$

$$
+ \frac{(n-9)\dots(n-13)}{1.2.3.4.5.6}\left\{\frac{6}{2^5.4} + \frac{5.3}{2^4.3^2}\right\}
$$

$$
\left. + \frac{(n-9)\dots(n-14)}{1.2.3.4.5.6.7}\left\{\frac{7}{2^6.3}\right\} + \frac{(n-9)\dots(n-15)}{1.2.3.4.5.6.7.8}\left\{\frac{1}{2^8}\right\}\right],
$$

$$
\left.\begin{array}{c}\dfrac{\Delta^{n-9}0^n}{\underline{\underline{n-9}}}\\ P_9\end{array}\right\} = n(n-1)\dots(n-9)\left[\frac{1}{10} + \frac{n-10}{1.2}\left\{\frac{2}{2.9} + \frac{2}{3.8} + \frac{2}{4.7} + \frac{2}{5.6}\right\}\right.
$$

$$
+ \frac{(n-10)(n-11)}{1.2.3}\left\{\frac{3}{2^2.8} + \frac{3.2}{2.3.7} + \frac{3.2}{2.4.6} + \frac{3}{2.5^2}\right.
$$

$$
\left. + \frac{3}{3^2.6} + \frac{3.2}{3.4.5} + \frac{1}{4^3}\right\}
$$

$$
+ \frac{(n-10)\dots(n-12)}{1.2.3.4}\left\{\frac{4}{2^3.7} + \frac{4.3}{2^2.3.6} + \frac{4.3}{2^2.4.5}\right.
$$

$$
\left. + \frac{4.3}{2.3^2.5} + \frac{4.3}{2.3.4^2} + \frac{4}{3^3.4}\right\}
$$

$$
+ \frac{(n-10)\dots(n-13)}{1.2.3.4.5}\left\{\frac{5}{2^4.6} + \frac{5.4}{2^3.3.5} + \frac{5.2}{2^3.4^2} + \frac{5.3.2}{2^2.3^2.4} + \frac{5}{2.3^4}\right\}
$$

$$
+ \frac{(n-10)\dots(n-14)}{1.2.3.4.5.6}\left\{\frac{6}{2^5.5} + \frac{6.5}{2^4.3.4} + \frac{5.4}{2^3.3^3}\right\}
$$

$$
+ \frac{(n-10)\dots(n-15)}{1.2.3.4.5.6.7}\left\{\frac{7}{2^6.4} + \frac{7.3}{2^5.3^2}\right\}
$$

$$
\left. + \frac{(n-10)\dots(n-16)}{1.2.3.4.5.6.7.8}\left\{\frac{8}{2^7.3}\right\} + \frac{(n-10)\dots(n-17)}{1.2.3.4.5.6.7.8.9}\left\{\frac{1}{2^9}\right\}\right].
$$

In taking the values of $P_1, P_2, \ldots P_9$ from this table, it is to be remembered that the *visible* numbers only are to be employed; in fact, $\overline{3}$ is to be read as if it were simply 3, $\overline{4}$ is to be read simply 4, and so on; in taking, however, the values of $\Delta^{n-1}0^n, \Delta^{n-2}0^n, \ldots \Delta^{n-9}0^n$, $\overline{3}, \overline{4}$, &c. are to be interpreted as standing for the factorials 1.2.3, 1.2.3.4, and so with the rest.

In obtaining these expressions, or in continuing the table, which may be easily done to any length, it will be found convenient to write the denominators of the fractions first, and then to supply the numerators by the common formula for variations.

The aspect of the foregoing table at once suggests a way to make palpable the identity between the expressions here given and those usually employed for $\Delta^m 0^n$.

$\Delta^{n-1}0^n, \Delta^{n-2}0^n$, &c., as here given, are nothing but certain terms selected from the expansions of $(n-1)^n, (n-2)^n$, &c., considered as polynomials of respectively $n-1$, $n-2$, &c. terms. The rule of selection is this; in the polynomial $(n-i)^n = \{1+1+1+\&c.\}^n$ replace the successive units by the quantities $a_1, a_2, \ldots a_{n-i}$, select the terms which each contain every one of these quantities, and then replace $a_1, a_2, \ldots a_{n-i}$ by units.

For example

$$\Delta^5 0^{10} = \overline{10}\left[\frac{5}{1.1.1.1.\overline{6}} + \frac{5.4}{1.2}\left\{\frac{2}{1.1.1.\overline{2}.\overline{5}} + \frac{2}{1.1.1.\overline{3}.\overline{4}}\right\}\right.$$

$$+\frac{5.4.3}{1.2.3}\left\{\frac{3}{1.1.\overline{2}^2.4} + \frac{3}{1.1.\overline{2}.3^2}\right\} + \frac{5.4.3.2}{1.2.3.4}\left\{\frac{4}{1.2^3.3}\right\}$$

$$\left. + \frac{5.4.3.2.1}{1.2.3.4.5}\left\{\frac{1}{\overline{2}^5}\right\}\right];$$

if we supply to the denominators within the brackets the complement of units, which makes the sum of the factors in each equal to 10, and consider each unit in the denominator as indicating one of the quantities a_1, a_2, a_3, &c. entering linearly, and each number as indicating the entrance of one of them in the corresponding power, we see that

$$\frac{5}{1.1.1.1.\overline{6}} = \Sigma\left\{a_1 a_2 a_3 a_4 \frac{a_5^6}{\overline{6}}\right\}_1, \quad \frac{5.4}{1.2}\left\{\frac{2}{1.1.1.\overline{2}.\overline{5}}\right\} = \Sigma\left\{a_1 a_2 a_3 \frac{a_4^2}{\overline{2}}\frac{a_5^5}{\overline{5}}\right\}_1,$$

and so on with the rest of the terms; but these symmetric

functions multiplied by $\overline{10}$ constitute the terms sought in the expansion of $(a_1+a_2+a_3+a_4+a_5)^{10}$. These terms, however, may be evidently expressed as follows:

$$(a_1+a_2+a_3+a_4+a_5)^{10}-\Sigma(a_1+a_2+a_3+a_4)^{10}+\Sigma(a_1+a_2+a_3)^{10}$$
$$-\Sigma(a_1+a_2)^{10}+\Sigma a_1^{10}.$$

The first Σ plainly contains five terms due to the successive omission of each suffix, the second contains as many terms as there are combinations in pairs of five quantities, or ten, and so with the rest; replacing, therefore, a_1, a_2, &c. with units, we obtain

$$5^{10}-5.4^{10}+10.3^{10}-10.2^{10}+5.1^{10},$$

the familiar expression for $\Delta^5 0^n$.

This way of regarding $\Delta^m 0^n$, which has been suggested by the table, may be synthetically established as follows:

$$\Delta^m 0^n = \{\varepsilon^{D_x}-1\}^m 0^n = \{(\varepsilon^{D_x}-1)^m x^n\}_0.$$

Now it is well known that

$$\{\phi(D_x)\,x^n\}_0 = \{D_x^{\,n}\phi(x)\}_0;$$

(for a theorem which includes this see Vol. IV., p. 89) therefore

$$\{(\varepsilon^{D_x}-1)^m x^n\}_0 = \{D_x^{\,n}(\varepsilon^x-1)^m\}_0.$$

Let $D_x = D_1+D_2+\dots D_m$, where $D_1, D_2, \dots D_m$ operate each on but one of the factors of $(\varepsilon^x-1)^m$, and if we suppose $(D_1+D_2\dots+D_m)^n$ to be expanded polynomially according to the extension of Leibnitz's theorem, we see that those terms only will remain after the substitution of zero for x, in which all the symbols $D_1, D_2, \dots D_m$ occur together, for they only will be without factors of the form $(\varepsilon^x-1)^p$.

Before proceeding to reduce the expressions in the table, I must recur to an expression for $\Delta^m 0^n$, which has been communicated to me by Dr. Hargreave, and which is of signal use in abridging the labour of reduction. The expression and its proof, as given by Dr. Hargreave, are as follows:

$$\frac{\Delta^m 0^n}{1.2\dots m} = \Sigma\left[(m+1)\,\Sigma\{(m+2)\dots\Sigma(n-1)\}\,\Sigma n\right],$$

where the Σ denotes finite integration; proof,

$$\Delta^m 0^{m+1} = m\Delta^m 0^m + m(m-1)\,\Delta^{m-1}0^{m-1} + m(m-1)(m-2)\,\Delta^{m-2}0^{m-2},$$

but $$\Delta^m 0^m = 1.2\dots m;$$

therefore

$$\Delta^m 0^{m+1} = 1.2\ldots m\,\{m + (m-1) + (m-2) + \ldots + 1\}$$
$$= 1.2\ldots m\,\Sigma\,(m+1),$$

$$\Delta^m 0^{m+2} = m\Delta^m 0^{m+1} + m\,(m-1)\,\Delta^{m-1}0^m$$
$$+ m\,(m-1)\,(m-2)\,\Delta^{m-2}0^{m-1} + \&c.,$$

or

$$\Delta^m 0^{m+2} = 1.2\ldots m\,\{m\Sigma(m+1) + (m-1)\Sigma(m) + (m-2)\Sigma(m-1) + \&c.\}$$
$$= 1.2\ldots m\Sigma\,\{(m+1)\,\Sigma\,(m+2)\,;$$

similarly,

$$\Delta^m 0^{m+3} = 1.2\ldots m\;\Sigma\,[(m+1)\,\Sigma\,\{(m+2)\,\Sigma\,(m+3)\}]\,;$$

and, generally,

$$\Delta^m 0^{m+p} = 1.2\ldots m\Sigma\,\{(m+1)\,\Sigma\,[(m+2)\ldots\Sigma\,\{(m+p-1)\,\Sigma\,(m+p)\}]\}.$$

Dr. Hargreave has directed my attention to the fact that this expression has been substantially anticipated by Mr. Jeffery (see vol. IV., p. 370) who states the theorem thus: $\dfrac{\Delta^m 0^{m+p}}{1.2\ldots m}$ is equal to the sum of the homogeneous products of the degree p, including powers of the successive numbers $1, 2, \ldots m+p-1$. It is, however, when expressed by finite integrals that the theorem is most useful for calculating the values of $\Delta^m 0^n$. The mechanical rule of partitions enables us to write down, with very little thought, the *factorial* expression of $\Delta^{n-p}0^n$ for any given value of p, and when we have condensed the coefficients of the factorials, it only remains to multiply by $n-p$ and take the finite integral in order to obtain $\Delta^{n-p-1}0^n$.

A comparison of the two expressions for $\dfrac{\Delta^m 0^n}{1.2\ldots m}$ leads to the theorem that

The sum of the homogeneous products, to the degree p (powers included), of the successive numbers $1.2\ldots n-1$ is equal to the sum of such homogeneous permutations to the degree $n-p$ of the reciprocals of the factorials $\overline{1}.\overline{2}\ldots\overline{n-p+1}$ as make by the addition of their visible numbers the sum n.

I now proceed to reduce the expressions in the table for $\dfrac{\Delta^m 0^n}{1.2\ldots m}$. It is plain that the terms within the brackets may be expressed as a function of n increased or diminished by a certain number, and that by selecting this number properly

the coefficients of the function may be much simplified. I have found that the coefficients are much reduced by taking for this number $n-m-1$ and *diminishing* n by it, $n-p+1$. I denote by the general symbol z for the sake of brevity.

$$\frac{\Delta^n 0^n}{\underline{n}} = 1,$$

$$\frac{\Delta^{n-1}0^n}{\underline{n-1}} = \frac{n(n-1)}{2},$$

$$\frac{\Delta^{n-2}0^n}{\underline{n-2}} = \frac{n(n-1)(n-2)}{2.4.3}\{3z-2\},$$

$$\frac{\Delta^{n-3}0^n}{\underline{n-3}} = \frac{n(n-1)\ldots(n-3)}{2.4.6}\{z^2-z\},$$

$$\frac{\Delta^{n-4}0^n}{\underline{n-4}} = \frac{n(n-1)\ldots(n-4)}{2.4.6.8.15}\{15z^3-15z^2-10z+8\},$$

$$\frac{\Delta^{n-5}0^n}{\underline{n-5}} = \frac{n(n-1)\ldots(n-5)}{2.4.6.8.10.3}\{3z^4-2z^3-7z^2+6z\},$$

$$\frac{\Delta^{n-6}0^n}{\underline{n-6}} = \frac{n(n-1)\ldots(n-6)}{2.4.6.8.10.12.63}\{63z^5-315z^3+224z^2+140z-96\},$$

$$\frac{\Delta^{n-7}0^n}{\underline{n-7}} = \frac{n(n-1)\ldots(n-7)}{2.4\ldots14.9}\{9z^6+9z^5-75z^4+23z^3+114z^2-80z\},$$

$$\frac{\Delta^{n-8}0^n}{\underline{n-8}} = \frac{n(n-1)\ldots(n-8)}{2.4\ldots16.135}\{5(27z^7+63z^6-315z^5-147z^4$$
$$+1064z^3-564z^2)-16(121z-72)\},$$

$$\frac{\Delta^{n-9}0^n}{\underline{n-9}} = \frac{n(n-1)\ldots(n-9)}{2.4\ldots18.15}\{15z^8+60z^7-355z^6-92z^5$$
$$+1285z^4-108z^3-1788z^2+784z\},$$

$$\frac{\Delta^{-10}0^n}{\underline{n-10}} = \frac{n(n-1)\ldots(n-10)}{2.4\ldots20.99}\{11(9z^9+54z^8-126z^7-588z^6$$
$$+128z^5+1166z^4-4012z^3+1672z^2+1312z)-7680\},$$

$$\frac{\Delta^{n-11}0^n}{\underline{n-11}} = \frac{n(n-1)\ldots(n-11)}{24\ldots.22.9}\{9z^{10}+75z^9-90z^8-1146z^7+1505z^6$$
$$+5179z^5-10496z^4-1036z^3+12912z^2-6912z\}.$$

In order to facilitate further calculation by Dr. Hargreave's formula I give $\dfrac{\Delta^{n-12}0^n}{n-12}$ in its unreduced factorial form

$$\frac{\Delta^{n-12}0^n}{1.2\ldots n-12} = \frac{n\ldots n-12}{2.4\ldots 22.9}\left\{\frac{3}{8}(n-13)\ldots(n-23)+33(n-13)\ldots(n-22)\right.$$
$$+1155(n-13)\ldots(n-21)+20834(n-13)\ldots(n-20)$$
$$+210100(n-13)\ldots(n-19)+1204368(n-13)\ldots(n-18)$$
$$+\frac{34451120}{9}(n-13)\ldots(n-17)+6308192(n-13)\ldots(n-16)$$
$$+4739504(n-13)\ldots(n-15)+\frac{6336256}{5}(n-13)\ldots(n-14)$$
$$\left.+\frac{483072}{7}(n-13)+\frac{1536}{13}\right\}.$$

Since σ_n or the sum the n^{th} powers of 1, 2, 3, $(r-1)$ is expressed thus:

$$\sigma_n = \Sigma\left\{\frac{\Delta^{n-p}0^n}{1.2\ldots(n-p+1)}\, r(r-1)(r-n+p)\right\},$$

(see No. 25) calculating some of the lower values of $\Delta^{n-p}0^n$, and reducing we find

$$\sigma_1 = \frac{r(r-1)}{2},$$

$$\sigma_2 = \frac{r(r-1)(2r-1)}{6},$$

$$\sigma_3 = \frac{r^2(r-1)^2}{4} = \sigma_1^2,$$

$$\sigma_4 = \frac{r(r-1)(2r-1)}{5.6}\{3r(r-1)-1\} = \frac{\sigma_2}{5}\{6\sigma_1-1\},$$

$$\sigma_5 = \frac{r^2(r-1)^2\{2r(r-1)-1\}}{3.4} = \frac{\sigma_1^2}{3}\{4\sigma_1-1\},$$

$$\sigma_6 = \frac{r(r-1)(2r-1)}{6.7}\{3r(r-1)[r(r-1)-1]+1\} = \frac{\sigma_2}{7}\{6\sigma_1(2\sigma_1-1)+1\},$$

$$\sigma_7 = \frac{r^2(r-1)^2}{4.6}\{r(r-1)[3r(r-1)-4]+2\} = \frac{\sigma_1^2}{3}\{2\sigma_1(3\sigma_1-2)+1\}.$$

Dr. Hargreave's expression in finite integrals of $\Delta^m 0^n$ suggests the expediency of seeking an analogous formula in

the case of the kindred functions P_1, P_2, &c., and in this there is no difficulty.

$P_2^{(n-1)}$ or the sum of the combinations in pairs of $1, 2 \ldots (n-1)$ is

$$(n-1)\,\Sigma\,(n-1)+(n-2)\,\Sigma\,(n-2)+\&c.,$$

or $\quad P_2^{(n-1)}=\Sigma\,(n\Sigma n)$, similarly $P_3^{(n-1)}=\Sigma\,(n\,(\Sigma\,(n\Sigma n)))$,

and generally omitting parentheses and considering that each Σ operates on all that comes after it,

$$P_p^{(n-1)}=\Sigma n\,\Sigma n \ldots \Sigma n \text{ to } p \text{ factors.}$$

A comparison of this expression with those given in the table leads to the following theorem:

The sum of the combinations, p together, of the numbers $1, 2, n-1$ is equal to the sum of such permutations and powers of their reciprocals, $n-p$ together, as make the sum of the denominators in each term equal to n.

The factorial expression for $P_p^{(n-1)}$ having been written by the rule of partitions for a given value of p and the coefficients of the factorials reduced $P_{p+1}^{(n-1)}$, $P_{p+2}^{(n-1)}$, &c. can be easily got by finite integration; subjoined are the values of these functions from $p=1$ to $p=8$ in a somewhat reduced form

$$P_1=\frac{n\,(n-1)}{2},$$

$$P_2=\frac{n\,(n-1)\,(n-2)\,(3n-1)}{2.4.3},$$

$$P_3=\frac{n\ldots(n-3)}{2.4.6}\,n\,(n-1),$$

$$P_4=\frac{n\,(n-1)\ldots(n-4)}{2.4.6.8.15}\,\{5\,(n-2)\,(3n^2+1)+12\},$$

$$P_5=\frac{n\ldots(n-5)}{2.4.6.8.10.3}\,\{(n-2)\,(3n-1)-4\}\,n\,(n-1),$$

$$P_6=\frac{n\ldots(n-6)}{2.4.6.8.10.12.63}\,\{7(n-1)(9n^2[n(n-4)+1]+2(11n+8))+96\},$$

$$P_7=\frac{n\ldots(n-7)}{2.4\ldots14.9}\,\{n(n-1)((n-2)\,[3(n+1)\{3n(n-5)+8\}-8]+48)\},$$

$$P_8=\frac{n\ldots(n-8)}{2.4\ldots16.135}\,\{5n^2((n-1)\{(n-3)[(n-1)(9(n-5)(3n+2)-65)-8] \\ -46\}+180)+4\,(101n+38\}.$$

These expressions suggest some relations between the functions P_1, P_2, ... &c.; in P_4, for instance,

$$5(n-2)(3n^2+1)+12=5(n-2)\{(3n-1)n+n+1\}+12,$$

or $$=5\{n(n-2)(3n-1)+5n(n-1)-2\}+12;$$

therefore

$$P_4=\frac{n\ldots(n-4)}{2.4.6.8.15}\left\{2.3.4.5\,\frac{P_2}{n-1}+10P+2\right\},$$

likewise

$$P_5=\frac{n\ldots(n-5)}{2.4\ldots10.3}\{2.3.4P_2-2.4P_3\},$$

$$P_6=\frac{n\ldots(n-6)}{2.4\ldots12.63}\left\{7\left(9n\left[16P_1^2-\frac{P_2}{n-2}\right]+44P_1+16(n-1)\right)+96\right\},$$

$$P_7=\frac{n(n-1)(n-7)}{2.4\ldots149}\left\{144P_2P_1+576\,\frac{P_3}{n-3}+96(3n-2)P_2-384P_2\right.$$
$$\left.+8n(n-1)(n-2)+48n(n-1)\right\}.$$

In this paper I have made less reference to the labours of previous writers on the same subject than perhaps I ought; it was not, however, until the paper was nearly completed that I became aware how much had been contributed to the *Journal* on this subject, especially by Mr. Jeffery. It was owing to his having seen a note, by this last named writer, on Staudt's Theorem, in the No. for May, 1863, that Dr. Hargreave was deterred from publishing a proof of it in which the expression of $\Delta^m 0^n$ by finite integrals was obtained and employed to prove the divisibility of $\Delta^m 0^n$ by $1.2\ldots m$.

Brighton Vale, Monkstown,
Co. Dublin,
May 15, 1865.

P.S. May 12, 1866. I could wish that the setting of the little relic which this paper contains had fallen into more skilful hands. I had thought to have shewn the paper, when published, to Dr. Hargreave, but now—"Fungor inani munere."

ON THE FORMULÆ WHICH CONNECT THE FOCI OF A CONIC SUBJECT TO THREE CONDITIONS.

By W. S. BURNSIDE, B.A.

FIRST, take the case of a conic circumscribing a triangle, and let the equations of its sides be $x=0$, $y=0$, $z=0$; this being so, the tangential equation of this conic is of the form

$$l^2\lambda^2 + m^2\mu^2 + n^2\nu^2 - 2mn\,\mu\nu - 2nl\,\nu\lambda - 2lm\,\lambda\mu = 0,$$

or $\Sigma_1 = 0$. Now applying the theory for the determination of the foci of a conic, as laid down by Dr. Salmon in his *Conic Sections*, (Chap. XVIII.), representing by $\Omega = 0$,

$$\lambda^2 + \mu + \nu^2 - 2\mu\nu\cos A - 2\nu\lambda\cos B - 2\lambda\mu\cos C = 0,$$

and by Σ,

$$A\lambda^2 + B\mu^2 + C\nu^2 + 2F\mu\nu + 2G\nu\lambda + 2H\lambda\mu,$$

$\Omega + k\Sigma_1$ denotes a system of conics confocal with Σ_1. And when k is determined so that $\Omega + k\Sigma_1$ represents a pair of points, these points are foci of the system. For convenience and generality, I substitute Σ for Ω, and then the points given by $\Sigma + k\Sigma_1$ are two opposite vertices of the quadrilateral formed by the common tangents to Σ and Σ_1, and it is easy to return at any stage to the consideration of the foci by making $A=B=C=1$, $F=-\cos A$, $G=-\cos B$, $H=-\cos C$.

Hence incorporating k with l, m, n, I can identify $\Sigma - \Sigma_1$ with $(\lambda x + \mu y + \nu z)(\lambda x_1 + \mu y_1 + \nu z_1)$, and comparing coefficients on both sides of this identity—

$$\begin{aligned} xx_1 &= A - l^2, & yz_1 + y_1z &= 2(F + mn),\\ yy_1 &= B - m^2, & zx_1 + z_1x &= 2(G + nl),\\ zz_1 &= C - n^2, & xy_1 + x_1y &= 2(H + lm), \end{aligned}$$

eliminating x_1, y_1, z_1 from the second set of equations by means of the first set, we shall have

$$\begin{aligned} (ny + mz)^2 &= Cy^2 - 2Fyz + Bz^2,\\ (lz + nx)^2 &= Az^2 - 2Gzx + Cx^2,\\ (mx + ly)^2 &= Bx^2 - 2Hxy + Ay^2. \end{aligned}$$

Now $Cy^2 - 2Fyz + Bz^2$ is the equation of the tangents drawn to the conic $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$ from the point y, z, denoting it by P, and the similar quantities by Q and R,

$$\begin{aligned} ny + mz &= \sqrt{(P)} & 2lyz &= -x\sqrt{(P)} + y\sqrt{(Q)} + z\sqrt{(R)}, \\ lz + nx &= \sqrt{(Q)} \text{ which give } & 2mzx &= x\sqrt{(P)} - y\sqrt{(Q)} + z\sqrt{(R)}, \\ mx + ly &= \sqrt{(R)} & 2nxy &= x\sqrt{(P)} + y\sqrt{(Q)} - z\sqrt{(R)}. \end{aligned}$$

Also, from the former, equations

$$yz_1 + y_1z : zx_1 + z_1x : xy_1 + x_1y :: F + mn : G + nl : H + lm;$$

eliminating l, m, n, by means of the latter equations the last proportion becomes

$$yz_1 + y_1z : zx_1 + z_1x : xy_1 + x_1y :: y^2z^2\{\sqrt{(QR)} - Dwx - Ayz\}$$
$$: z^2x^2\{\sqrt{(RP)} - Dwy - Bzx\} : x^2y^2\{\sqrt{(PQ)} - Dwz - Cxy\},$$

where $$Fx + Gy + Hz + Dw = 0.$$

From this proportion we can determine the locus described by the point (xyz), when the locus of the point $(x_1y_1z_1)$ is given —, of course all the radicals in the above formulæ are susceptible of double sign; giving four positions for the point $(x_1y_1z_1)$ for each position of the point (xyz); it was in fact from observing this *a priori*, that I was tempted to seek the above relation, depending on three radicals.

It may be interesting to apply the foregoing results to determine the locus described by one focus when the other moves on a right line.

Let the equation of this line be $x + y + z$, or δ, and write α, β, γ for

$$-x + y + z,\ x - y + z,\ x + y - z;$$

then the locus assumes the form

$$A\alpha y^2z^2 + B\beta z^2x^2 + C\gamma x^2y^2 + D\delta xyzw$$
$$= \alpha yz\sqrt{(QR)} + \beta zx\sqrt{(RP)} + \gamma xy\sqrt{(PQ)}.$$

Squaring both sides, the part of the result free of radicals, is

$$\alpha^2y^2z^2(QR - A^2y^2z^2) + \beta^2z^2x^2(RP - B^2z^2x^2)$$
$$+ \gamma^2x^2y^2(PQ - C^2x^2y^2) - D^2\delta^2x^2y^2z^2w^2$$
$$- 2x^2y^2z^2(BC\beta\gamma x^2 + CA\gamma\alpha y^2 + AB\alpha\beta z^2)$$
$$- 2D\delta xyzw(A\alpha y^2z^2 + B\beta z^2x^2 + C\gamma x^2y^2).$$

Now denoting

$$BCx^2 + CAy^2 + ABz^2 - 2AFyz - 2BGzx - 2CHxy$$

by W, then

$$QR - A^2y^2z^2 = x^2 W + 2ADxyzw + 4(GH + AF)x^2yz,$$
$$RP - B^2z^2x^2 = y^2 W + 2BDxyzw + 4(HF + BG)xy^2z,$$
$$PQ - C^2x^2y^2 = z^2 W + 2CDxyzw + 4(FG + CH)xyz^2,$$

substituting these values, and putting U for

$$(\alpha^2 + \beta^2 + \gamma^2)W - D\delta^2w^2 - 2\{BC\beta\gamma x^2 + CA\gamma\alpha y^2 + AB\alpha\beta z^2$$
$$- 2(GH + AF)\alpha^2yz - 2(HF + BG)\beta^2zx - (FG + GH)\gamma^2xy\},$$

we get

$$x^2y^2z^2U + 2Dxyzw\{A\alpha y^2z^2(\alpha - \delta) + B\beta z^2x^2(\beta - \delta) + C\gamma x^2y^2(\gamma - \delta)\},$$

which is equivalent to

$$x^2y^2z^2\{U - 4Dw(A\alpha yz + B\beta zx + C\gamma xy)\},$$

as the reduced form for the part free of radicals, when the equation has been once squared. Writing $Sx^2y^2z^2$ for the last result, and squaring both sides of the equation again, the locus assumes the form

$$x^4y^4z^4S^2 = 4PQRx^2y^2z^2\{x^2\beta^2\gamma^2P + y^2\gamma^2\alpha^2Q + z^2\alpha^2\beta^2R$$
$$+ 2\alpha\beta\gamma(A\alpha y^2z^2 + B\beta z^2x^2 + C\gamma x^2y^2 + D\delta xyzw\}.$$

We now proceed to show that the right side of this equation is *further* divisible by $x^2y^2z^2$.

Replacing P, Q, R, and Dw, by their values in terms of xyz,

$$x^2\beta^2\gamma^2P + y^2\gamma^2\alpha^2Q + z^2\alpha^2\beta^2R$$
$$+ 2\alpha\beta\gamma(A\alpha y^2z^2 + B\beta z^2x^2 + C\gamma x^2y^2 + D\delta xyzw)$$

becomes

$$A\alpha^2y^2z^2(\beta + \gamma)^2 + B\beta^2z^2x^2(\gamma + \alpha)^2 + C\gamma^2x^2y^2(\alpha + \beta)^2$$
$$- 2xyz\{F\beta\gamma x(\beta\gamma + \alpha\delta) + G\gamma\alpha y(\gamma\alpha + \beta\delta) + H\alpha\beta z(\alpha\beta + \gamma\delta)\},$$

and hence we have by substituting for α, β, γ, δ, in terms of xyz, $(\beta\gamma + \alpha\delta = 4yz$, &c.), the reduced value

$$4x^2y^2z^2(A\alpha^2 + B\beta^2 + C\gamma^2 - 2F\beta\gamma - 2G\gamma\alpha - 2H\alpha\beta),$$

finally substituting this quantity in the former equation, and

dividing off by $x^4y^4z^4$, the equation of the locus assumes the form

$$S^2 = 16PQR\,(A\alpha^2 + B\beta^2 + C\gamma^2 - 2F\beta\gamma - 2G\gamma\alpha - 2H\alpha\beta),$$

where $S = (\alpha^2 + \beta^2 + \gamma^2)\,W - 2\,\{BC\beta\gamma x^2 + CA\gamma\alpha y^2 + AB\alpha\beta z^2$

$- 2\,(GH + AF)\,\alpha^2 yz - 2\,(HF + BG)\,\beta^2 zx - 2\,(FG + CH)\,\gamma^2 xy\}$

$- D^2\delta^2 w^2 - 4Dw\,(A\alpha yz + B\beta zx + C\gamma xy)$,

and the tangential equation of the circular points at infinity, $(A, B, C, F, G, H),\ (\lambda, \mu, \nu)^2 = 0$.

It is plain from the inspection of the equation that the vertices of the triangle are multiple foci of the locus.

Secondly, take the case of a self-conjugate triangle with regard to a conic being given.

Let the equations of the sides of this triangle be $x = 0$, $y = 0$, $z = 0$, then the tangential equation of the conic is of the form $L\lambda^2 + M\mu^2 + N\nu^2 = 0$, or $\Sigma_1 = 0$.

Now identifying as before $\Sigma + k\Sigma_1$ with

$$(\lambda x + \mu y + \nu z)\ \ (\lambda x_1 + \mu y_1 + \nu z_1)$$

we get the proportion

$$yz_1 + y_1 z : z_1 x + zx_1 : xy_1 + x_1 y :: F : G : H;$$

from which we have

$$x_1 : y_1 : z_1 :: x(-Fx + Gy + Hz) : y(Fx - Gy + Hz) : z(Fx + Gy - Hz)$$

as the relation connecting the foci of the conic.

It is easy, in this case, to find the locus described by one focus, when the other describes *any* curve; take for example the case, when one focus moves on a right line and let its equation be $lx + my + nz = 0$; then from the proportion

$$x_1 : y_1 : z_1 :: x\,(-x\cos A + y\cos B + z\cos C)$$
$$: y\,(x\cos A - y\cos B + z\cos C) : z\,(x\cos A + y\cos B - z\cos C)$$

(restoring F, G, H, their values), we have as the equation of the locus

$$lx(-x\cos A + y\cos B + z\cos C) + my(x\cos A - y\cos B + z\cos C)$$
$$+ nz\,(x\cos A + y\cos B - z\cos C) = 0,$$

a conic, passing through the feet of the perpendiculars of the triangle, and the harmonic conjugates of the points where the line $lx + my + nz$ meets the sides.

Thirdly, let a conic be inscribed in a triangle, which being taken as the triangle of reference, the tangential equation

of this conic is of the form $L\mu\nu + M\nu\lambda + N\lambda\mu$; identifying as before

$$A\lambda^2 + B\mu^2 + C\nu + 2F\mu\nu + 2G\nu\lambda + 2H\lambda\mu + k(L\mu\nu + M\nu\lambda + N\lambda\mu)$$

with $$(\lambda x + \mu y + \nu z)\ (\lambda x_1 + \mu y_1 + \nu z_1).$$

I find the simple relation

$$xx_1 : yy_1 : zz_1 :: A : B : C$$

connecting two opposite vertices of the quadrilateral formed by the common tangents to Σ and Σ_1. The case when Σ represents a pair of points has been treated by Prof. Cayley in this *Journal.* (See Vol. IV. p. 131).

31, Trinity College, Dublin,
15 *March*, 1865.

NOTE ON TENSION.

By W. H. Besant, M.A., St. John's College, Cambridge.

In considering questions relating to the tension of flexible and inextensible surfaces, it is essential to observe that the tension along any line on the surface is not in general perpendicular to that line. It is assumed that this is the case when the line is a line of curvature, and when the forces in action on the surface consist only of normal forces, such, for instance, as those produced by the action of fluids. In such cases the equation

$$\frac{t}{\rho} + \frac{t'}{\rho'} = p$$

is established, p being the fluid pressure; but this equation does not hold when the directions of t and t' are not those of principal curvature.

The general equations of equilibrium of a flexible surface are investigated in the fourth volume of this *Journal*, and the relation thence deduced between the tensions at any point in two directions at right angles, and the tangential actions at the same point and in the same directions. It is also mentioned that the same relation may be obtained more directly, and the following is the proof alluded to:

D 2

Take any point O (fig. 1) on the surface, and take two directions OA, OB at right angles to each other.

Let t, t' be the tensions in these directions, and T', T the tangential actions in the same directions.

Oz being the normal at O, draw four planes parallel to, and very near to, the normal planes AOz, BOz, cutting the surface in CD, DE, EF, FC.

Then, ultimately, the tangential actions, $T.CD$ and $T.EF$ on CD and EF are equal and opposite, as are also those on ED and CF.

Hence, by taking moments about Oz, it appears that

$$T = T'.$$

Let the tangent lines to OA, OB be axes of x and y, and let $(x, 0, z)$ be coordinates of A.

The value of $\frac{dz}{dx}$ at $A = \frac{d^2z}{dx^2}x + \frac{1}{2}\frac{d^3z}{dx^3}x^2 + \ldots$, the expressions $\frac{d^2z}{dx^2}, \ldots$ being the values of the differential coefficients at O. So

$$\frac{dz}{dy} \text{ at } A = \frac{d^2z}{dx\,dy}x + \ldots.$$

The equations of the tangent line to OA at A are

$$X = x, \quad Z - z = sx.Y,$$

where
$$s = \frac{d^2z}{dx\,dy},$$

and its direction-cosines are

$$0, \quad \frac{1}{\sqrt{(1+s^2x^2)}}, \quad \frac{sx}{\sqrt{(1+s^2x^2)}}.$$

Similarly, the direction-cosines of the tangent at B are

$$\frac{1}{\sqrt{(1+s^2y^2)}}, \quad 0, \quad \frac{sy}{\sqrt{(1+s^2y^2)}}.$$

Taking p for the normal pressure at O, and resolving the forces on $CDEF$ parallel to Oz, we obtain ultimately

$$2t.CD\frac{OA}{\rho} + 2t'.DE\frac{OB}{\rho'} + T.CD.sx - T.EF(-sx)$$
$$+ T.ED.sy - T.CF(-sy) = p.CD.DE,$$

or
$$\frac{t}{\rho} + \frac{t'}{\rho'} + 2T\frac{d^2z}{dx\,dy} = p \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

If, instead of a fluid pressure, the force on $CDEF$ be

$$\sigma\varepsilon . CD . DE . f,$$

where σ is the density and ε the thickness of the surface, the equation is

$$\frac{t}{\rho} + \frac{t'}{\rho'} + 2T\frac{d^2z}{dx\,dy} = \sigma\varepsilon f.$$

Recurring to the case of fluid or normal pressure, and assuming that in such cases the tension along a line of curvature is perpendicular to that line, we have the equation

$$\frac{t}{\rho} + \frac{t'}{\rho'} = p,$$

and from this we can deduce the equation (1) in the following manner:

Taking OA, OB (fig. 2) as the directions of principal curvature, let OAB be a small triangular portion of the surface, T the tension perpendicular to AB, and τ the tangential action on AB. Then, if $OBA = \theta$, resolving perpendicular to AB,

$$T . AB = t . OB \cos\theta + t' . OA \sin\theta,$$

or $$T = t\cos^2\theta + t'\sin^2\theta \text{..................(2)},$$

whence we observe that the sum of the tensions at O in any two directions at right angles is the same.

Also $$\tau . AB + t . OB \sin\theta = t' . OA \cos\theta,$$

or $$\tau = (t' - t)\sin\theta\cos\theta \text{..................(3)}.$$

Let T, T' (fig. 3) be the tensions in two directions OP, OQ at right angles to each other, and τ the tangential action parallel to OP or OQ.

Then, taking r, r' as the radii of curvature in these directions, we obtain

$$\begin{aligned}\frac{T}{r} + \frac{T'}{r'} &= t\left(\frac{1}{r}\cos^2\theta + \frac{1}{r'}\sin^2\theta\right) \\ &\quad + t'\left(\frac{1}{r}\sin^2\theta + \frac{1}{r'}\cos^2\theta\right) \\ &= t\left\{\frac{1}{\rho}(\cos^4\theta + \sin^4\theta) + \frac{2}{\rho'}\sin^2\theta\cos^2\theta\right\} \\ &\quad + t'\left\{\frac{1}{\rho'}(\cos^4\theta + \sin^4\theta) + \frac{2}{\rho}\sin^2\theta\cos^2\theta\right\},\end{aligned}$$

since $$\frac{1}{r} = \frac{1}{\rho} \cos^2\theta + \frac{1}{\rho'} \sin^2\theta,$$

or $$\frac{T}{r} + \frac{T'}{r'} = \frac{t}{\rho} + \frac{t'}{\rho'} - 2(t - t') \sin^2\theta \cos^2\theta \left(\frac{1}{\rho} - \frac{1}{\rho'}\right)$$

$$= p + \tau \sin 2\theta \left(\frac{1}{\rho} - \frac{1}{\rho'}\right).$$

But $$\frac{1}{r} - \frac{1}{r'} = \cos 2\theta \left(\frac{1}{\rho} - \frac{1}{\rho'}\right),$$

and, if OP, OQ be axes of x and y, the angle θ is given by the equation

$$\tan 2\theta = \frac{2\dfrac{d^2z}{dx\,dy}}{\dfrac{d^2z}{dy^2} - \dfrac{d^2z}{dx^2}} = 2\frac{d^2z}{dx\,dy} \div \left(\frac{1}{r'} - \frac{1}{r}\right).$$

Hence by substitution there results the equation

$$\frac{T}{r} + \frac{T'}{r'} + 2\tau \frac{d^2z}{dx\,dy} = p.$$

It may be remarked that the equations (2) and (3) are true even when the forces which act on the surface are not under the restriction of being normal forces. For the forces in action on any element of the surface are proportional to its area, or to the square of the linear dimension, and therefore ultimately vanish in comparison with the tensions, which are proportional to the linear dimension, *i.e.* to the length of a side of the element.

Further, by taking an element in the form of a small right-angled triangle, it can be shewn that there are, in any case, two directions at right angles to each other, in which the tangential action is evanescent.

ON FORMULÆ OF CURVATURE IN TERMS OF TRILINEAR COORDINATES.

By WILLIAM WALTON, M.A., Trinity College.

I KNOW not whether expressions, in terms of trilinear coordinates, for the parameters of the circle of curvature at any point of a curve, have been given by any writer on algebraic geometry. As such formulæ would be occasionally useful, perhaps, if they be not already known, this essay on curvature may be of some little interest to students.

I will proceed in the first place to investigate an expression for the radius of curvature, and will afterwards determine formulæ for the determination of the position of the centre of the osculating circle.

If θ be the angle between the lines (l, m, n), (l', m', n'), then*

$\tan\theta$

$$=\frac{(mn'-nm')\sin A+(nl'-ln')\sin B+(lm'-ml')\sin C}{ll'+mm'+nn'-(mn'+nm')\cos A-(nl'+ln')\cos B-(lm'+ml')\cos C}:$$

hence, if $d\phi$ denote the angle between two consecutive tangents of the curve $F(\alpha, \beta, \gamma)=0$, that is, between the two lines

$$(U,\ V,\ W),\ (U+dU,\ V+dV,\ W+dW),$$

where $$U=\frac{dF}{d\alpha},\quad V=\frac{dF}{d\beta},\quad W=\frac{dF}{d\gamma},$$

then $d\phi$

$$=\frac{(VdW-WdV)\sin A+(WdU-UdW)\sin B+(UdV-VdU)\sin C}{U^2+V^2+W^2-2VW\cos A-2WU\cos B-2UV\cos C}$$

$$=\frac{2\Delta}{abc}\cdot\frac{a(VdW-WdV)+b(WdU-UdW)+c(UdV-VdU)}{U^2+V^2+W^2-2VW\cos A-2WU\cos B-2UV\cos C}.$$

* Ferrers: *Trilinear Coordinates*, p. 22.

Differentiating the equations

$$a\alpha + b\beta + c\gamma = 2\Delta, \quad F = 0,$$

we have

$$a d\alpha + b d\beta + c d\gamma = 0,$$

$$U d\alpha + V d\beta + W d\gamma = 0;$$

and therefore

$$d\alpha : d\beta : d\gamma :: L : M : N,$$

where $L = Vc - Wb, \quad M = Wa - Uc, \quad N = Ub - Va.$

Let

$$\frac{d^2F}{d\alpha^2} = u, \quad \frac{d^2F}{d\beta^2} = v, \quad \frac{d^2F}{d\gamma^2} = w,$$

$$\frac{d^2F}{d\beta\, d\gamma} = u', \quad \frac{d^2F}{d\gamma\, d\alpha} = v', \quad \frac{d^2F}{d\alpha\, d\beta} = w'.$$

Then

$$V dW - W dV = V(v' d\alpha + u' d\beta + w d\gamma) - W(u' d\gamma + w' d\alpha + v d\beta),$$

and therefore, putting

$$d\alpha = \lambda L, \quad d\beta = \lambda M, \quad d\gamma = \lambda N,$$

we have

$$\frac{1}{\lambda}(V dW - W dV) = V(Lv' + Mu' + Nw) - W(Nu' + Lw' + Mv):$$

similarly

$$\frac{1}{\lambda}(W dU - U dW) = W(Mw' + Nv' + Lu) - U(Lv' + Mu' + Nw),$$

$$\frac{1}{\lambda}(U dV - V dU) = U(Nu' + Lw' + Mv) - V(Mw' + Nv' + Lu).$$

Hence

$$\frac{1}{\lambda}\{a(V dW - W dV) + b(W dU - U dW) + c(U dV - V dU)\}$$

$$= Lu(Wb - Vc) + Mv(Uc - Wa) + Nw(Va - Ub)$$

$$+ u'\{M(Va - Ub) + N(Uc - Wa)\}$$

$$+ v'\{N(Wb - Vc) + L(Va - Ub)\}$$

$$+ w'\{L(Uc - Wa) + M(Wb - Vc)\}$$

$$= -L^2u - M^2v - N^2w - 2u'MN - 2v'NL - 2w'LM.$$

Moreover

$$abc\,(U^2+V^2+W^2-2VW\cos A-2WU\cos B-2UV\cos C)$$
$$=abc\,(U^2+V^2+W^2)$$
$$\quad -aVW(b^2+c^2-a^2)-bWU(c^2+a^2-b^2)-cUV(a^2+b^2-c^2)$$
$$=-a\,(Wa-Uc)\,(Ub-Va)$$
$$\quad -b\,(Ub-Va)\,(Vc-Wb)$$
$$\quad -c\,(Vc-Wb)\,(Wa-Uc)$$
$$=-aMN-bNL-cLM.$$

Hence

$$d\phi=2\lambda\Delta.\frac{L^2u+M^2v+N^2w+2u'MN+2v'NL+2w'LM}{aMN+bNL+cLM}.$$

Again, if ds be an element of the arc of the curve, we have, by the formula for the distance between two points,*

$$ds^2=-\frac{abc}{4\Delta^2}.(ad\beta\,d\gamma+bd\gamma\,d\alpha+cd\alpha\,d\beta)$$
$$=-\frac{abc}{4\Delta^2}.\lambda^2.(aMN+bNL+cLM).$$

But, if ρ be the radius of curvature, $\rho=\frac{ds}{d\phi}$: hence

$$\rho^2=-\frac{abc}{16\Delta^4}.\frac{(aMN+bNL+cLM)^3}{(L^2u+M^2v+N^2w+2u'MN+2v'NL+2w'LM)^2},$$

or, supposing $\frac{d}{d\alpha}$, $\frac{d}{d\beta}$, $\frac{d}{d\gamma}$, not to operate on L, M, N,

$$\rho^2=-\frac{abc}{16\Delta^4}.\frac{(aMN+bNL+cLM)^3}{\left\{\left(L\frac{d}{d\alpha}+M\frac{d}{d\beta}+N\frac{d}{d\gamma}\right)^2F\right\}^2}.$$

Having investigated an expression for the radius of the circle of curvature, I proceed now to the determination of the coordinates of its centre.

The equation to the tangent to the curve, at the point (α, β, γ), being

$$\alpha'U+\beta'V+\gamma'W=0\ \ldots\ldots\ldots\ldots\ldots\ldots(1),$$

let the equation to the normal be

$$l\alpha'+m\beta'+n\gamma'=0\ \ldots\ldots\ldots\ldots\ldots\ldots(2).$$

* Ferrers: *Trilinear Coordinates*, p. 6.

Since the lines (1) and (2) are perpendicular to each other, we know that*

$$lU+mV+nW=(nV+mW)\cos A$$
$$+(lW+nU)\cos B+(mU+lV)\cos C\ldots(3).$$

Since (α, β, γ) is a point in the normal,

$$l\alpha+m\beta+n\gamma=0 \ldots\ldots\ldots\ldots\ldots\ldots (4).$$

From (2) and (4) we see that

$$l:m:n::\beta'\gamma-\beta\gamma':\gamma'\alpha-\gamma\alpha':\alpha'\beta-\alpha\beta' \ldots (5).$$

From (3) and (5) we obtain

$$\begin{aligned}(\beta'\gamma-\beta\gamma')\,U&+(\gamma'\alpha-\gamma\alpha')\,V+(\alpha'\beta-\alpha\beta')\,W\\ &=\cos A\,\{(\alpha'\beta-\alpha\beta')\,V+(\gamma'\alpha-\gamma\alpha')\,W\}\\ &+\cos B\,\{(\beta'\gamma-\beta\gamma')\,W+(\alpha'\beta-\alpha\beta')\,U\}\\ &+\cos C\,\{(\gamma'\alpha-\gamma\alpha')\,U+(\beta'\gamma-\beta\gamma')\,V\},\end{aligned}$$

and therefore

$$\begin{aligned}&(\beta'\gamma-\beta\gamma')\,(W\cos B+V\cos C-U)\\ +&(\gamma'\alpha-\gamma\alpha')\,(U\cos C+W\cos A-V)\\ +&(\alpha'\beta-\alpha\beta')\,(V\cos A+U\cos B-W)=0.\end{aligned}$$

This equation and the equation

$$(\alpha'-\alpha)\,a+(\beta'-\beta)\,b+(\gamma'-\gamma)\,c=0,$$

are satisfied identically by the equations

$$\frac{\alpha-\alpha'}{W\cos B+V\cos C-U}$$
$$=\frac{\beta-\beta'}{U\cos C+W\cos A-V}$$
$$=\frac{\gamma-\gamma'}{V\cos A+U\cos B-W},$$

which are therefore the equations to the normal at the point (α, β, γ).

Let
$$\begin{aligned}W\cos B+V\cos C-U&=P,\\ U\cos C+W\cos A-V&=Q,\\ V\cos A+U\cos B-W&=R;\end{aligned}$$

and let $\quad \alpha'-\alpha=\lambda_1P,\ \beta'-\beta=\lambda_1Q,\ \gamma'-\gamma=\lambda_1R \ldots\ldots(6):$

* Ferrers: *Trilinear Coordinates*, p. 20.

differentiating on the hypothesis that α', β', γ', are constant, that is, considering $(\alpha', \beta', \gamma')$ to be the intersection of two consecutive normals, we have

$$d\alpha + Pd\lambda_1 + \lambda_1 dP = 0,$$

$$d\beta + Qd\lambda_1 + \lambda_1 dQ = 0,$$

$$d\gamma + Rd\lambda_1 + \lambda_1 dR = 0,$$

and thence we see that

$$\lambda_1 = \frac{Rd\beta - Qd\gamma}{RdQ - QdR} = \frac{Pd\gamma - Rd\alpha}{PdR - RdP} = \frac{Qd\alpha - Pd\beta}{QdP - PdQ}$$

$$= \frac{(Q-R)\,d\alpha + (R-P)\,d\beta + (P-Q)\,d\gamma}{(Q-R)\,dP + (R-P)\,dQ + (P-Q)\,dR} \quad \ldots\ldots(7).$$

Now

$$(Q-R)\,dP = (\cos B . dW + \cos C . dV - dU) \times \{(\cos C - \cos B)\,U + (1 + \cos A)\,(W - V)\},$$

$$(R-P)\,dQ = (\cos C . dU + \cos A . dW - dV) \times \{(\cos A - \cos C)\,V + (1 + \cos B)\,(U - W)\},$$

$$(P-Q)\,dR = (\cos A . dV + \cos B . dU - dW) \times \{(\cos B - \cos A)\,W + (1 + \cos C)\,(V - U)\}.$$

Adding together these three equations, we get

$$(Q-R)\,dP + (R-P)\,dQ + (P-Q)\,dR$$

$$= (WdV - VdW)\,(1 + \cos A)\,(1 + \cos B + \cos C - \cos A)$$

$$+ (UdW - WdU)\,(1 + \cos B)\,(1 + \cos C + \cos A - \cos B)$$

$$+ (VdU - UdV)\,(1 + \cos C)\,(1 + \cos A + \cos B - \cos C).$$

Now, by substituting for $\cos A$, $\cos B$, $\cos C$, their values in terms of the sides of the triangle of reference, we may easily ascertain that, $a + b + c$ being represented by $2s$,

$$(1 + \cos A)\,(1 + \cos B + \cos C - \cos A) = \frac{8s\Delta^2}{a^2b^2c^2} . a,$$

$$(1 + \cos B)\,(1 + \cos C + \cos A - \cos B) = \frac{8s\Delta^2}{a^2b^2c^2} . b,$$

$$(1 + \cos C)\,(1 + \cos A + \cos B - \cos C) = \frac{8s\Delta^2}{a^2b^2c^2} . c.$$

Hence $$(Q-R)\,dP+(R-P)\,dQ+(P-Q)\,dR$$
$$=\frac{8s\Delta^2}{a^2b^2c^2}.(LdU+MdV+NdW).$$

But
$$dU=\lambda\,(uL+w'M+v'N),$$
$$dV=\lambda\,(vM+u'N+w'L),$$
$$dW=\lambda\,(wN+v'L+u'M):$$

hence $$(Q-R)\,dP+(R-P)\,dQ+(P-Q)\,dR$$
$$=\lambda.\frac{8s\Delta^2}{a^2b^2c^2}.(uL^2+vM^2+wN^2+2u'MN+2v'NL+2w'LM)$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots(8).$$

Again, $$(Q-R)\,d\alpha+(R-P)\,d\beta+(P-Q)\,d\gamma$$
$$=\lambda L\,\{(\cos C-\cos B)\,U+(1+\cos A)\,(W-V)\}$$
$$+\lambda M\,\{(\cos A-\cos C)\,V+(1+\cos B)\,(U-W)\}$$
$$+\lambda N\,\{(\cos B-\cos A)\,W+(1+\cos C)\,(V-U)\}.$$

Substituting for $\cos A$, $\cos B$, $\cos C$, their values in terms of a, b, c, and performing obvious algebraical processes, we shall get, for the coefficients of λL, λM, λN, the respective values

$$\frac{2s\,(s-a)}{abc}\,(M+N),\quad \frac{2s\,(s-b)}{abc}\,(N+L),\quad \frac{2s\,(s-c)}{abc}\,(L+M):$$

hence $$(Q-R)\,d\alpha+(R-P)\,d\beta+(P-Q)\,d\gamma$$
$$=\lambda.\frac{2s}{abc}.(aMN+bNL+cLM)\ldots\ldots\ldots\ldots(9).$$

From (6), (7), (8), (9), we have, for the trilinear coordinates of the centre of the circle of curvature,

$$\frac{\alpha-\alpha'}{W\cos B+V\cos C-U}$$
$$=\frac{\beta-\beta'}{U\cos C+W\cos A-V}$$
$$=\frac{\gamma-\gamma'}{V\cos A+U\cos B-W}$$
$$=\frac{abc}{4\Delta^2}.\frac{aMN+bNL+cLM}{\left(L\frac{d}{d\alpha}+M\frac{d}{d\beta}+N\frac{d}{d\gamma}\right)^2F}.$$

The formula which we obtained for the radius of curvature may be deduced as a corollary from the formulæ for the centre of the circle of curvature.

Replacing the cosines of the angles of the triangle of reference by their expressions in terms of the sides, the formulæ become

$$\frac{\alpha-\alpha'}{(b^2-c^2)L+a(bM-cN)}$$
$$=\frac{\beta-\beta'}{(c^2-a^2)M+b(cN-aL)}$$
$$=\frac{\gamma-\gamma'}{(a^2-b^2)N+c(aL-bM)}$$
$$=\frac{1}{8\Delta^2}\cdot\frac{aMN+bNL+cLM}{\left(L\frac{d}{d\alpha}+M\frac{d}{d\beta}+N\frac{d}{d\gamma}\right)^2 F}.$$

Then, the radius of curvature being equal to the distance between the two points (α, β, γ), $(\alpha', \beta', \gamma')$, we have

$$\frac{\rho^2}{Q}=a\{(c^2-a^2)M+b(cN-aL)\}\{(a^2-b^2)N+c(aL-bM)\}$$
$$+b\{(a^2-b^2)N+c(aL-bM)\}\{(b^2-c^2)L+a(bM-cN)\}$$
$$+c\{(b^2-c^2)L+a(bM-cN)\}\{(c^2-a^2)M+b(cN-aL)\},$$

where $$Q=-\frac{abc}{256\Delta^6}\cdot\left\{\frac{aMN+bNL+cLM}{\left(L\frac{d}{d\alpha}+M\frac{d}{d\beta}+N\frac{d}{d\gamma}\right)^2 F}\right\}^2.$$

In the expression for $\frac{\rho^2}{Q}$, the coefficients of L^2, M^2, N^2, are respectively equal to $-a^3bc$, $-b^3ca$, $-c^3ab$; and, Δ denoting as usual the area of the triangle of reference, the coefficients of MN, NL, LM, are respectively equal to

$$2a(8\Delta^2-b^2c^2),\quad 2b(8\Delta^2-c^2a^2),\quad 2c(8\Delta^2-a^2b^2):$$

hence $$\frac{\rho^2}{Q}=16\Delta^2(aMN+bNL+cLM)$$
$$-abc(La+Mb+Nc)^2,$$

and therefore, $La+Mb+Nc$ being identically zero,

$$\rho^2=-\frac{abc}{16\Delta^4}\cdot\frac{(aMN+bNL+cLM)^3}{\left\{\left(L\frac{d}{d\alpha}+M\frac{d}{d\beta}+N\frac{d}{d\gamma}\right)^2 F\right\}^2}.$$

July 26, 1864.

Addition by Prof. Cayley.

In my memoir "On the Conic of Five-Pointic Contact," *Philosophical Transactions*, 1859, I gave incidentally (p. 378) the *equation* of the circle of curvature at any point of a plane curve, viz. writing (α, β, γ), F, instead of my (x, y, z), U, and putting therefore

$$D = X\frac{d}{d\alpha} + Y\frac{d}{d\beta} + Z\frac{d}{d\gamma},$$

then taking (X, Y, Z) as current coordinates, the equation of the circle of curvature at the point (α, β, γ) of the curve $F(\alpha, \beta, \gamma) = 0$, of the order m, is

$$\begin{vmatrix} XDF, & YDF, & ZDF, & D^2F \\ \alpha\ , & \beta\ , & \gamma\ , & 2(m-2) \\ \alpha_1\ , & \beta_1\ , & \gamma_1\ , & \left(\dfrac{D^2F}{DF}\right)_1 \\ \alpha_2\ , & \beta_2\ , & \gamma_2\ , & \left(\dfrac{D^2F}{DF}\right)_2 \end{vmatrix} = 0,$$

where $(\alpha_1, \beta_1, \gamma_1)$, $(\alpha_2, \beta_2, \gamma_2)$ are the coordinates of the circular points at infinity, and $\left(\dfrac{D^2F}{DF}\right)_1 \left(\dfrac{D^2F}{DF}\right)_2$ are the values of $\left(\dfrac{D^2F}{DF}\right)$ when these coordinates respectively are substituted for the current coordinates (X, Y, Z). In the system of coordinates employed by Mr. Walton, the circular points at infinity are given as the intersections of the line $a\alpha + b\beta + c\gamma = 0$ with the circle $a\beta\gamma + b\gamma\alpha + c\alpha\beta = 0$, that is, we have for $(\alpha_1, \beta_1, \gamma_1)$ and $(\alpha_2, \beta_2, \gamma_2)$ the values $\left(\dfrac{1}{a}, \dfrac{\omega}{b}, \dfrac{\omega^2}{c}\right)$ and $\left(\dfrac{1}{a}, \dfrac{\omega^2}{b}, \dfrac{\omega}{c}\right)$ respectively, ω being an imaginary cube root of unity. Substituting these values, my result should be equivalent to those obtained by Mr. Walton, but the form is so different that I have not attempted to make the comparison.

DEMONSTRATIONS OF SOME KNOWN GEOMETRICAL THEOREMS.

By W. F. Walker, M.A.

(I). From a point Q (fig. 4) on the circle circumscribing a triangle ABC; perpendiculars QD, QE, QF, are let fall on the sides, meeting them in the points D, E, F, respectively. These three points lie on one right line, which is equidistant from the point Q, and from the polar centre of the triangle.

Dem. Let the line QP joining Q with the polar centre (P) be bisected in M, which therefore lies on the nine-point circle of the triangle, and let the line MD connecting M with D, the foot of the perpendicular from Q on BC, meet CA in E; and AB in F; then it will easily appear that QE and QF are perpendicular to CA and AB respectively. For if APX be the perpendicular from A on BC, and if U be the middle point of AP; the points M, U, X lie on the nine-point circle; and since $MU = \frac{1}{2}AQ$, the angle QCE is equal to the angle MXU, and therefore to the angle QDE; from whence it follows, that the quadrilateral $QCDE$ is circumscribable, and therefore that QEC is a right angle. Similarly QFB is a right angle.

Since then the interval QP is bisected by the right line DEF, these two points are equidistant from that line. Q. E. D.

(II). The circumscribing, nine-points, and polar circles of a triangle ABC, form a coaxal system.

Lemma. Ax, By, Cz being the perpendiculars from the vertices on the opposite sides; let yz meet BC in U; then will the line UP (P being the polar centre) be perpendicular to $A\alpha$; where α is the middle point of BC.

For, let AU (fig. 5) meet the circumscribing circle in L, and join LP, the rectangle $AU.LU = BU.CU = yU.zU$; whence it follows that the quadrilateral $AyzL$ is circumscribable, now the quadrilateral $AyzP$ is likewise circumscribable; hence the angle $ALy =$ the angle $Azy =$ the angle APy; and the quadrilateral $APLy$ is circumscribable, and therefore ALP is a right angle. If then D is diametrically opposite to A on the circumscribing circle, PL passes through D; also $P\alpha$ passes through D, and therefore the four points

D, a; P, L lie on one right line. Now in the triangle AaU; Ax and aL perpendiculars from two vertices on the opposite sides meet in P, and therefore UP is perpendicular to the remaining side Aa. Q.E.D.

Dem. Rectangle $UX.Ua$ = rectangle $UA.UL$ = rectangle $UP.UQ$, where Q is the intersection of Aa and

$$UP = UP^2 - UP.PQ = UP^2 - PA.PX = UP^2 - (\text{polar radius})^2;$$

now $UP^2 -$ (polar radius)^2 = square of tangent from U to polar circle; therefore the tangents from U to the three circles are equal, and therefore, &c.

It may be observed, that the formula stated in Mr. Townsend's *Modern Geometry*, Vol. I., p. 255, as connecting the radii of three coaxal circles, whose centres are A, B, C; and radii AR, BS, CT respectively, viz.

$$\frac{AR^2}{AB.AC} + \frac{BS^2}{BC.BA} + \frac{CT^2}{CA.CB} = 1,$$

supplies an easy demonstration of the same theorem.

(III). If I (fig. 6) be the centre of the circle inscribed in a triangle ABC; and P be the polar centre, then will

$$PI^2 = 2r^2 + \rho^2,$$

where r is the radius of inscribed circle and ρ the radius of polar circle.

In the demonstration the following relations are assumed, all of which are elementary and easily proved.

(1) The point I being the mean centre of the three points A, B, C; for the multiples a, b, c (representing the lengths of the sides), by the fundamental property of the mean centre (Townsend's *Modern Geometry*, Vol. I., p. 130, Art. 98),

$$(a+b+c).PI^2 = (a.PA^2 + b.PB^2 + c.PC^2) - (a.IA^2 + b.IB^2 + c.IC^2).$$

(2) $$PA + PB + PC = 2(R + r).$$

(3) $$a.IA^2 + b.IB^2 + c.IC^2 = a.b.c.$$

(4) $$\frac{a.b.c}{a+b+c} = 2R.r.$$

Dem. Let the perpendiculars from the vertices on the opposite sides be Ax, By, Cz intersecting in P, and observe that $a.Ax = b.By = c.Cz = 2S$, where S = area of triangle, also $AP.Px = BP.Py = CP.Pz = (-\rho^2)$; (supposing all the angles to be acute).

Now by (1),

$$(a+b+c)PI^2=(a.PA^2+b.PB^2+c.PC^2)-(a.IA^2+b.IB^2+c.IC^2),$$

and $\quad a.PA^2=a.AX.AP-a.AP.PX=2S.AP+a.\rho^2;$

similarly for others, hence

$$a.PA^2+...=2S(AP+BP+CP)$$
$$+(a+b+c).S^2=2S(QR+2r)+(a+b+c).S^2,$$

hence from (3) and (4), after dividing both sides of (1) by $(a.b.c)$

$$PI^2=r(2R+2r)-S^2-2R.r=2r^2+S^2. \quad \text{Q.E.D.}$$

Cor. Assuming the analogous, but more easily proved expression for the distance between P and O, where O is the centre of the circumscribing circle, viz. $PO^2=R^2+2\rho^2$; we can readily prove that if N is the centre of the nine-point circle, $NI=r-\dfrac{R}{2}$; and therefore that the nine-point circle touches the inscribed circle.

For, considering that in the triangle OPI, N is the middle point of the side opposite to I; therefore

$$2NI^2=OI^2+PI^2-\frac{OP^2}{2},$$

$$2NI^2=R^2-2Rr+2r^2+S^2-\tfrac{1}{2}[R^2+2S^2],$$

$$NI^2=\frac{R^2}{4}-Rr+r^2=\left(r-\frac{R}{2}\right)^2;$$

therefore $\quad NI=r-\dfrac{R}{2}. \quad$ Q.E.D.

The proofs of the corresponding expressions in reference to the centres of the exscribed circles differ in no respect, but in a slight modification of sign.

(IV). Let P (fig. 7) be an arbitrary point within a triangle ABC; let a parallel to BC through P meet AB and AC in x, x'; a parallel to CA meet BC and BA in y, y'; a parallel to AB meet CA and CB in z, z'; and let xPx' intersect the circumscribing circle in L and M; it is required to prove that

$$Px.Px'+Py.Py'+Pz.Pz'=PL.PM.$$

Lemma. If A, B, C, D are four points taken in order on one right line; it is easily proved that

$$AC.BD=AD.BC+AB.CD.$$

$$Dem.\quad Py.Py' = Cx'.Az = Cx'.Ax'.\frac{Az}{Ax'} = Lx'.Mx'.\frac{Px}{xx'},$$

$$Pz.Pz' = Ay'.Bx = Ax.Bx.\frac{Ay'}{Ax} = Lx.Mx.\frac{Px'}{xx'};$$

therefore $$Py.Py' + Pz.Pz' = \frac{Lx'.Mx'.Px + Lx.Mx.Px'}{xx'}$$

$$= \frac{xx'.Mx'.Px + Lx\,(Mx'.Px + Mx.Px')}{xx'}.$$

Now, by Lemma, $Mx'.Px + Mx.Px' = MP.xx'$, hence evidently

$$Px.Px' + Py.Py' + Pz.Pz' = Px.Px' + Px.Mx' + MP.Lx$$
$$= Px.PM + PM.Lx = LP.PM;$$

therefore $Px.Px' + Py.Py' + Pz.Pz' = PL.PM$. Q.E.D.

NOTE. The demonstrations of theorems III. and IV., as given above, were obtained and exhibited to pupils in the spring of the year 1860. It may be desirable to mention this, as other demonstrations of these theorems have appeared in the *Quarterly Journal*, at periods subsequent to that date.

9, Trinity College, Dublin,
July, 1865.

NOTE ON THE GEOMETRY OF THE TRIANGLE.

By JOHN GRIFFITHS, M.A.

LET D, E, F denote the feet of the perpendiculars AD, BE, CF of any given triangle ABC; through each of the vertices A, B, C let a parallel be drawn to the opposite side of the triangle DEF, and $D'E'F'$ be the triangle formed by these three lines:

Then, if H denote the axis of homology of the triangles ABC, $D'E'F'$, the circle which circumscribes DEF (nine-point circle) passes through the two points common to the polar circles of the four triangles that can be formed from the lines BC, CA, AB, and H.

If ABC be taken as the triangle of reference, the equations to the sides of the triangle $D'E'F'$ are, as shewn in the last Number of this *Journal*,

$$\frac{\beta}{b} + \frac{\gamma}{c} = 0,\quad \frac{\gamma}{c} + \frac{\alpha}{a} = 0,\quad \frac{\alpha}{a} + \frac{\beta}{b} = 0,$$

and that of the line H will, therefore, be

$$\frac{\alpha}{a}+\frac{\beta}{b}+\frac{\gamma}{c}=0.$$

(The sides $E'F'$, $F'D'$, $D'E'$ are, in fact, the tangents at A, B, C to the circle which passes through these three points).

Hence, if h_1, h_2, h_3 denote the points where H meets the sides BC, CA, AB, it is easily found that the polar circles of the four triangles ABC, Ah_2h_3, Bh_3h_1, Ch_1h_2 are given by the equations

$$a\cos A\,.\,\alpha^2+b\cos B\,.\,\beta^2+c\cos C\,.\,\gamma^2=0,$$

$$a\cos A\,.\,\alpha^2+b\cos B\,.\,\beta^2+c\cos C\,.\,\gamma^2 + \frac{2a^2}{bc}\cos A\,(a\beta\gamma+b\gamma\alpha+c\alpha\beta)=0,$$

$$a\cos A\,.\,\alpha^2+b\cos B\,.\,\beta^2+c\cos C\,.\,\gamma^2 + \frac{2b^2}{ca}\cos B\,(a\beta\gamma+b\gamma\alpha+c\alpha\beta)=0,$$

$$a\cos A\,.\,\alpha^2+b\cos B\,.\,\beta^2+c\cos C\,.\,\gamma^2 + \frac{2c^2}{ab}\cos C\,(a\beta\gamma+b\gamma\alpha+c\alpha\beta)=0:$$

also, that of the circle which passes through the points D, E, F is

$$a\cos A\,.\,\alpha^2+b\cos B\,.\,\beta^2+c\cos C\,.\,\gamma^2-(a\beta\gamma+b\gamma\alpha+c\alpha\beta)=0\,;$$

these five circles are, therefore, coaxal; and, of course, belong to the system noticed in Numbers 24 and 28 (pp. 359, 346) of the *Quarterly Journal.*

NOTE. The two points common to the system in question are not only the centres of the two equilateral hyperbolas which touch the sides of the triangle of reference and the line H, but also the foci of the two parabolas which pass through the points D', E', F', and the centre of homology of the triangles ABC, $D'E'F'$.

This and some other theorems connected with the geometry of the triangle I propose to prove in a future paper.

Jesus College, Oxford,
May 1st, 1866.

ON INTERPOLATION WITH REFERENCE TO DEVELOPMENT AND DIFFERENTIATION.

PART II.

(Continued from Vol. VII., p. 212.)

By SAMUEL ROBERTS, M.A.

1. In the previous part,* I have endeavoured to establish that the most convenient and natural generalization by interpolation of the Binomial Theorem, is expressed by the formula

$$(a+x)^m = \sum_{k=-\text{integer}\,\infty}^{k=+\text{integer}\,\infty} \frac{\Gamma m+1}{\Gamma p+k+1\,\Gamma m-p-k+1} x^{p+k} \ldots (a),$$

a formula subject, it is true, to the difficulties usually attending interpolation of form, but in which those difficulties do not appear to be exaggerated. In my processes, I hold fast to the "principle of the permanence of equivalent forms," for I can by no means admit that the principle has been invalidated by the exceptional arguments brought against it, and it seems to me, that Professor Kelland commits himself to too broad a statement when he says (*Trans. of the Royal Society, Edinburgh*, Vol. XIV., p. 570), "There can be no doubt that such a principle has no real existence, sanctioned as it is by the names of the greatest analysts." In the literal calculus, it is not unreasonable to assign a higher authority to "form" than to arithmetical value, a truth to which Dr. Hutton has given valuable testimony (*Tracts*, vol. I., Tract 7), and which he did much to establish by his remarkable researches on divergent series. It seems to me that every system of general development must, for the most part give divergent results, when the argument is real.

In persistently following out the consequences of a law of operation, I have been led to conclusions, relative to

* Since the former part was printed, I have met with a notice by M. E. Catalan, "Sur une application de la formule binome aux integrales Euleriennes" (*Comptes Rendus*, 1858). The close relation of this notice to the present subject is indicated by the title.

certain definite integrals, which conflict with received results. And since definite integration essentially involves the idea of value, those conclusions seem to require confirmation or explanation. They are, however, merely collateral to my main purpose, and may stand as suggestions rather than as proved results.

The weak point of the present theory is, of course, the case of m negative. The circumstance, that the series (g) is divergent, for $m =$ or < -1, corresponding to the change of Cauchy's integral, might lead us to consider the function $\Gamma m+1$ as infinite, whenever m is $=$ or < -1, while the functions in the denominator of the equivalent expression retain their extended meaning. It is, indeed, the, to some extent, arbitrary element of the generalization which justifies the extension of (a) to general negative values of m.

Thus, by the mode of generation adopted, we get, for the coefficient of x^{-m} in $(1+x)^{-m-1}$, the series

$$\left\{1+m+\frac{m.m+1}{1.2}+\ldots\right\}=(1-1)^{-m}=\infty.$$

But the formula gives

$$\frac{\Gamma -m}{\Gamma 1-m\,\Gamma 0}=0,$$

which is correct, since

$$(1+x)^{-m-1}=x^{-m-1}-(m+1)\,x^{-m-2}+\&c.$$

And it will be found, that the formula holds good by actual expansion, whenever $m+p$ (p the index of the argument) is a whole number. We have, then,

$$\text{co. of } x^p \text{ in } \frac{1}{\sin m\pi}\,\Sigma_{-\infty}^{+\infty}\frac{\Gamma m+p}{\Gamma 1+p\,\Gamma m}\sin(m+p)\,\pi x^p$$

$$=\text{co. of } x^p \text{ in } \frac{\sin(m+p)\,\pi}{\sin m\pi}\,\Sigma_{-\infty}^{+\infty}\frac{\Gamma m+p}{\Gamma 1+p\,\Gamma m}\,x^p;$$

when $m+p$ is a whole number, and the extension consists in taking $\sin(m+p)\,\pi$ as general, which we have already found to be so in the case of $(1+x)^0$. There is also another case, where the equivalence is independently true, viz. when $m+p=-m$.

I have been obliged to use as the base unit $(1+x)^0$ corresponding to $(1+x)^m$, or $(a+x)^0$ corresponding to $(a+x)^m$, in order to bring the processes within an available compass.

It is evident, that the simple substitute of $(C+x)^0$ for unity introduces an arbitrary constant.

2. The indeterminateness of the problem makes it necessary to lay down a rule, which must govern our application of the calculus of operations to the subject. The rule may be stated thus:

Equivalent development of symbols of quantity and symbols of operation must be formed according to the same laws.

In other words, we must not, when comparison is made, use one particular solution in the case of symbols of quantity, and another in the case of symbols of operation.

To obtain a value of $\left(\frac{d}{dx}\right)^n x^m$, I proceed thus:

That $\left(\frac{d}{dx}\right)^n x^m$ is of the form kx^{m-n}, k being independent of x, is generally agreed. We also have

$$\epsilon^{D_x}.x^m = (1+x)^m.$$

Comparing corresponding terms of the developments of the left and right hand members, we get

$$\frac{D_x^n}{\Gamma 1+n} x^m = \frac{\Gamma 1+m}{\Gamma 1+n\,\Gamma 1+m-n} x^{m-n},$$

or

$$D_x^n.x^m = \frac{\Gamma 1+m}{\Gamma 1+m-n} x^{m-n} \;\ldots\ldots\ldots\ldots\ldots\ldots (b),$$

which is the form of Euler, adopted by Peacock. It may be transformed, when advisable, by means of

$$\Gamma s\,\Gamma 1-s = \frac{\pi}{\sin s\pi}.$$

Thus are obtained

$$D_x^n.x^m = \frac{\sin(n-m)\,\pi}{\pi}\,\Gamma 1+m\,\Gamma n-m\,x^{m-n},$$

$$D_x^n.x^{-m} = \frac{\sin(n+m)\,\pi}{\sin m\pi}\,\frac{\Gamma m+n}{\Gamma m}\,x^{-m-n},$$

$$D_x^{-n}.x^m = \frac{\Gamma 1+m}{\Gamma 1+m+n}\,x^{m+n},$$

$$D^{-n}.x^{-m} = \frac{\sin(m-n)\,\pi}{\sin m\pi}\,\frac{\Gamma m-n}{\Gamma m}\,x^{n-m}.$$

These forms remain finite, except when the index of x is a negative integer, and that of D_x is fractional or a negative integer numerically = or > the index of x. The last is a well known result of ordinary integration, the effect of which, however, is avoided by aid of the complementary function of integration.

3. The theory of general differentiation obliges us to introduce complementary functions, not only when the index is a negative integer (in integration), but whenever the index is fractional.

M. Liouville pointed out, in the memoirs already alluded to, the necessity of taking such functions into account. As discovered by him, they appear so arbitrary as to render the result in a very high degree indefinite, and they are, in truth, so general that the issue of M. Liouville's reasoning is not to be wondered at.

To complete the formula (b), we must add to the right-hand member the general form of the function ϕx, which satisfies

$$D_x^{-n}.\phi x = 0.$$

Now, the formula (b) is evanescent, when m is fractional or a positive integer, and $n-m$ is a positive integer > 0. Hence we have

$$D_x^{-n}.x^{-n-k} = 0 \text{ (k a positive integer).}$$

And the complementary function may be written as

$$k_1x^{-n-1} + k_2x^{-n-2} + k_3x^{-n-3} + \ldots,$$

the series terminating only when $-n-3$, one of the indices, becomes cypher, in which case the last term is a constant. Also, if n is a positive integer, there is no complementary function, since the indices would be negative integers, which are not allowable.

If k_1, k_2, ... could be taken as infinite *ad libitum*, this form of function would be entirely arbitrary; since any function could be developed in such a manner. But, since

$$k_kD_x^{-n}x^{-n-k} = 0,$$

we must have k_k finite, or, at most, of an order of infinity inferior to that of the reciprocal of $D^{-n}x^{-n-k}$. The retention, therefore, of the series in the form deduced is generally advantageous, as will also appear from some uses we shall make of it.

4. Let us take into account the term $k_k x^{n-k}$ of the complementary function proper to

$$D_x^{-n}.x^{-m} = \frac{\pi}{\sin m\pi\,\Gamma m\,\Gamma 1+n-m}\,x^{n-m}.$$

$$\text{Then } D_x^{-n}.x^{-m} = \frac{\pi}{\sin m\pi\,\Gamma m\,\Gamma 1+n-m}\{x^{n-m} - a^{k-m}.x^{n-k}\} + C.F$$

$$= \frac{x^{k-m} - a^{k-m}}{\sin m\pi}\cdot\frac{\pi}{\Gamma m\,\Gamma 1+n-m}x^{n-k} + C.F.$$

Now, when $m = k$ (a positive integer > 0), we have

$$\frac{x^{k-m} - a^{k-m}}{\sin m\pi} = -\frac{\log\frac{x}{a}}{\pi\cos k\pi} = \pm\frac{\log\frac{x}{a}}{\pi},$$

according as k is odd or even.

Hence, finally,

$$D_x^{-n}.x^{-k} = \pm\frac{x^{n-k}\log\frac{x}{a}}{\Gamma k\,\Gamma 1+n-k} + C.F.$$

But it will be observed, that the expression can only assume this form when n is fractional or a positive integer = or > k; otherwise, there is no term of the complementary function of the form $k_k x^{n-k}$.

Thus far the general formula (b) has been directly applied, and I have obtained results consistent with themselves and with the rules of the differential and integral calculus.

5. The development, know as Leibnitz's theorem, and the modern extensions of it, may be made use of. Thus, the expression

$$D_x^n uv = uD_x^n v + nD_x uD_x^{n-1}v + \frac{n(n-1)}{1.2}D_x uD_x^{n-2} + \&c.....(c)$$

may be generalised. For, if we substitute for D_x the sum of two operators (say $D_a + D_b$), operating respectively on the coefficients of u and v, and equivalent in united results to D_x, we must have $D_x^n.uv = (D_a + D_b)^n u.v$, and the development ($c$) at once follows. By ($c$), we get

$$D_x^{-n}(uD_x^n v) = uv - nD_x^{-n}(D_x u.D^{n-1}v)$$

$$-\frac{n.n-1}{1.2}D^{-n}(D_x^2 u.D^{n-2}v) - \&c. \;.........(c'),$$

which is the formula for differentiation by parts.

As an example, let $D_x^{\,n} v = x^t$, or $v = \dfrac{\Gamma 1 + t}{\Gamma 1 + t + n} x^{t+n}$, $u = \log x$, then we have

$$D_x^{-n}.x^t \log x = \frac{\Gamma 1 + t}{\Gamma 1 + t + n} x^{t+n} \log x$$

$$- D_x^{-n}.\left\{\frac{n}{x} . \frac{x^{t+1}}{t+1} - \frac{n.n-1}{1.2}\left(-\frac{1}{x^2}\right)\frac{x^{t+2}}{t+1.t+2} + \&c.\right\}$$

$$= \frac{\Gamma 1 + t}{\Gamma 1 + t + n} x^{t+n} \left\{\log x - \frac{n^{t+1}}{t+1} + \frac{n.n-1}{1.2.t+1.t+2}\right.$$

$$\left. - \frac{n.n-1.n-2}{1.2.3.t+1.t+2.t+3}.2 + \&c.\right\} \ldots\ldots\ldots\ldots (d).$$

If t is, while $1 + t + n$ is not a negative integer, the expression becomes infinite, and a finite value can only be obtained by taking into further account the complementary function. If $1 + t + n$ is, while $1 + t$ is not, a negative integer, the expression has the factor 0, but, in this case, the process is founded on $v = 0.x^{t+n}$. I shall revert to this expression hereafter. The expressions obtained by different developments of the foregoing kind are not usually identical. For, in general, the complementary functions enter. This is well known to be the case in common integration. Differentiation to integer indices is free from this objection, since no complementary functions can be introduced. There is no need to dwell on these circumstances, since they equally occur in common integration, and are especially met with in the application of the calculus of operations to the solution of differential equations.

If the formula (d) is integrated on both sides, after writing it as

$$D_x^{-n} x^t \log x = M\{x^{t+n} \log x - N x^{t+n},$$

we have

$$D_x^{-n-1} x^t \log x = M\left\{\frac{x^{t+n+1}\log x}{1+t+n} - \frac{x^{t+n+1}}{(1+t+n)^2} - \frac{N x^{t+n+1}}{1+t+n}\right\};$$

that is to say,

$$D_x^{-n-1} x^t \log x = \frac{\Gamma 1 + t x^{t+n+1}}{\Gamma 2 + t + n}\left\{\log x - N - \frac{1}{1+t+n}\right\}.$$

Moreover, if $n = 0$, we have

$$D^{-1} x^t \log x = \frac{\Gamma 1 + t}{\Gamma 2 + t} x^{t+1}\left\{\log x - \frac{1}{1+t}\right\}$$

$$= \frac{x^{t+1}}{t+1}\left\{\log x - \frac{1}{1+t}\right\}.$$

Hence, if n is an integer,

$$D^{-n}x^t \log x = \frac{\Gamma 1+t}{\Gamma 1+t+n} x^{t+n} \left\{\log x - \frac{1}{1+t} - \frac{1}{2+t} \ldots - \frac{1}{n+t}\right\} \ldots\ldots(e).$$

In fact, when n is a positive integer,

$$\frac{n}{t+1} - \frac{n.n-1}{1.2.t+1.t+2} + \frac{n.n-1.n-2}{1.2.3.t+1.t+2}.2 - \&c.$$

$$= \frac{1}{1+t} + \frac{1}{2+t} + \ldots + \frac{1}{n+t}.$$

Thus the formula agrees with the known result of integration by parts.

6. I propose, now, to obtain an expression for $D_x^n \varepsilon^{ax}$.

Since
$$\varepsilon^{ax} = a^0x^0 + \frac{ax}{1} + \frac{a^2x^2}{1.2} + \ldots$$
$$+ \frac{a^{-1}x^{-1}}{\Gamma 0} + \frac{a^{-2}x^{-2}}{\Gamma -1} + \frac{a^{-3}x^{3}}{\Gamma -2} + \ldots,$$

we have
$$D_x^n \varepsilon^{ax} = a^n \left\{\frac{a^{-n}x^{-n}}{\Gamma 1-n} + \frac{a^{-n+1}x^{-n+1}}{\Gamma 2-n} + \frac{a^{-n+2}x^{-n+2}}{\Gamma 3-n} + \ldots \right.$$
$$\left. + \frac{a^{-n-1}x^{-n-1}}{\Gamma n} + \frac{a^{-n-2}x^{-n-2}}{\Gamma -n-1} + \frac{a^{-n-3}x^{-n-3}}{\Gamma -n-2} + \ldots\right\}$$
$$= a^n.\varepsilon^{ax} \ldots\ldots\ldots\ldots (f),$$

which is Leibnitz's form, obtained by him through the immediate generalization of the known form for integer operations.

The demonstration may be varied thus:

$$D_x^n \varepsilon^{ax} = D_x^n \left(1 + \frac{ax}{m}\right)^m (m = \infty)$$
$$= \frac{\Gamma 1+m}{\Gamma 1+m-n} \left(\frac{a}{m}\right)^n \left(1 + \frac{ax}{m}\right)^{m-n} (m = \infty)$$
$$= a^n.\varepsilon^{ax} \ldots\ldots\ldots\ldots (f),$$

as before, since $\dfrac{\Gamma 1+m}{\Gamma 1+m-n.m^n} (m = \infty) = 1.$

7. Circular functions remain to be considered. It is unnecessary, however, for me to dwell on these, because their differentials are obtained by transforming them into imaginary exponential functions and operating according to (f). Now, in regard to the exponential function ε^{ax}, the result just given coincides with that adopted by both Liouville and Kelland. I may therefore refer the reader to their memoirs for the treatment of such functions. As far as I know, the two forms of Euler and Leibnitz have not hitherto been actually reconciled.

The application, also, of the preceding formulæ to numerical examples is so direct, that I need not elaborate examples. For purposes of comparison, however, I give a few examples from Kelland numbered as in his memoirs.

(Part I., Sec. I.) Ex. 4. Find the value of $\dfrac{d^{\frac{1}{2}}x^{\frac{1}{2}}}{dx^{\frac{1}{2}}}$.

We have directly $$\frac{\Gamma 1 + \frac{1}{2}}{\Gamma 1} x^0 = \frac{\sqrt{(\pi)}}{2}.$$

(Part I., Sec. I.) Ex. 5. Find the value of $\dfrac{d^{\frac{1}{2}}x^{\frac{3}{2}}}{dx^{\frac{1}{2}}}$.

We have $$\frac{\Gamma 1 + \frac{3}{2}}{\Gamma 2} x = \tfrac{3}{2} . \tfrac{1}{2} \sqrt{(\pi)} . x.$$

(Part I., Sec. I.) Ex. 6. Find the value of $\dfrac{d^{\frac{3}{2}}x^{\frac{1}{2}}}{dx^{\frac{3}{2}}}$.

We have $$\frac{\Gamma 1 + \frac{1}{2}}{\Gamma 0} x^{-1} = 0.$$

(Part I., Sec. I.) Ex. 7. Find the value of $\dfrac{d^n x^n}{dx^n}$.

The result is $$\frac{\Gamma 1 + n}{\Gamma 1} x^0 = \Gamma 1 + n.$$

(Part I., Sec. I.) Ex. 8. Find the value of $\dfrac{d^{\frac{1}{2}}x}{dx^{\frac{1}{2}}}$.

The result is $$\frac{\Gamma 2}{\Gamma 2 - \frac{1}{2}} x^{\frac{1}{2}} = \frac{2x^{\frac{1}{2}}}{\sqrt{(\pi)}}.$$

Agreeably to Euler's conclusion. *Pet. Trans.* Vol. V. 1730.

(Part I., Sec. I.) Ex. 9. Find the value of $\dfrac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}$ (constant).

The formula gives $$\frac{\Gamma 1}{\Gamma 1 - \frac{1}{2}} x^{-\frac{1}{2}} = \frac{x^{-\frac{1}{2}}}{\sqrt{(\pi)}}.$$

Logarithmic Functions.

(Part I., Sec. II.) To find the differential of 0, the index of $\log x$. We have

$$\frac{d^{\frac{1}{2}} \log x}{dx^{\frac{1}{2}}} = \frac{x^{-\frac{1}{2}}}{\sqrt{(\pi)}} \left\{ \log x + \tfrac{1}{2} + \frac{1.3}{(1.2)^2 2^2} + \frac{1.3.5.2}{(1.2.3)^2 2^3} + \ldots \right\}.$$

(Part I., Sec. II.) Ex. 3. To find $\dfrac{d^{\frac{3}{2}} \log x}{dx^{\frac{3}{2}}}$.

The formula (d) gives

$$\frac{d^{\frac{3}{2}}}{dx^{\frac{3}{2}}} \log x = -\frac{x^{-\frac{3}{2}}}{2\sqrt{(\pi)}} \left\{ \log x + \tfrac{3}{2} + \frac{3.5}{(1.2)^2 2^2} + \frac{3.5.7.2}{(1.2.3)^3 2^3} + \ldots \right\}.$$

(Part I., Sec. II.) Ex. 3. To find $\dfrac{d^{\frac{3}{2}} x \log x}{dx^{\frac{3}{2}}}$.

The formula gives

$$\frac{d^{\frac{3}{2}}}{dx^{\frac{3}{2}}} x \log x = \frac{x^{-\frac{1}{2}}}{\sqrt{(\pi)}} \left\{ \log x + \frac{3}{2.2} + \frac{3.5}{1.2.2.3.2^2} + \frac{3.5.7.2}{1.2.3.2.3.4.2^3} + \ldots \right\}.$$

As to circular functions, I accept, as before stated, the solutions given by Professor Kelland. It must be remembered, that where fractional indices appear in a result, that result is multiple according to the number and arrangement of the roots. Thus $\dfrac{d^{\frac{1}{4}} \cos x}{dx^{\frac{1}{4}}}$ has four values and so on.

8. In formula (d), there has been retained within the brackets a series, which has generally an arithmetical value, and evidently terminates when n is a positive integer. In the foregoing examples also the series has been retained. But (δ) is susceptible of a more concise statement. It will be observed, that, for $t = 0$, the series is simply $\dfrac{d}{dn} \log \dfrac{1}{\Gamma 1 + n}$; and we may write

$$D_x^{-n} \log x = \frac{x^n}{\Gamma 1 + n} \left\{ \log x + \frac{d}{dn} \log \frac{1}{\Gamma 1 + n} \right\}$$

$$= x^n \left\{ \frac{1}{\Gamma 1 + n} \log x + \left(\frac{1}{\Gamma 1 + n} \right)' \right\}.$$

In the general expression, accepting according to previous reasoning, for the general function representing the quantity $(t+1.t+2.t+3\ldots t+n)$, the function $\dfrac{\Gamma t+n+1}{\Gamma 1+t}$, (see De Morgan's *Diff. and Int. Calculus*, p. 597), we have

$$D_x^{-n}x^t\log x = \frac{\Gamma 1+t}{\Gamma 1+t+n}\,x^{t+n}\left\{\log x - \frac{d}{dt}\log\frac{\Gamma 1+t+n}{\Gamma 1+t}\right\}$$

$$= x^{t+n}\left\{\frac{\Gamma 1+t}{\Gamma 1+t+n}\log x + \frac{d}{dt}\,\frac{\Gamma 1+t}{\Gamma 1+t+n}\right\}.$$

Now this form receives a remarkable confirmation, for because

$$D_x^{-n}x^t = \frac{\Gamma 1+t}{\Gamma 1+t+x}\,x^{t+n},$$

differentiation with regard to t gives

$$D_x^{-n}x^t\log x = \frac{\Gamma 1+t}{\Gamma 1+t+n}\,x^{t+n}\log x + x^{t+n}\frac{d}{dt}\,\frac{\Gamma 1+t}{\Gamma 1+t+n}\ \ldots\ (g).$$

If $t+n$ is a negative integer, the coefficient of the logarithmic term is evanescent, while that of x^{t+n} is not necessarily so. In fact, supposing that t is a positive integer and n a negative integer numerically $> t$, the second term is finite and gives the proper result of ordinary differentiation.

Differentiating p times with regard to t, we get

$$D_x^{-n}x^t(\log x)^p = \left(\frac{d}{dt}\right)^p\frac{\Gamma 1+t}{\Gamma 1+t+n}\,x^{t+n}\ \ldots\ldots.\ (h);$$

here p is integer, but the form suggests generalization to any value of p, and we only have to apply the formula (f). The foregoing formula is more general than (b) and includes it.

9. The expansion of functions by means of general differentiation has been but slightly treated by writers on the subject. The examples taken are of the simplest character, obviously chosen to illustrate the theory in a simple manner and not for the sake of the results.

I find such functions as $(x+h)(a-x-h)^{\frac{1}{2}}$, $\dfrac{(x+h)^2}{x+h-a}$, $\dfrac{1}{\sqrt{(1+x)}}$ according to powers of $\dfrac{1}{x}$. Since these forms depend on binomials, the application of the formula before given is direct. It is, indeed, only necessary to apply the general

binomial theorem (a). The results of Messrs. Greatheed and Kelland will be thus obtained with, as it appears to me, much greater ease. I confine myself, on this part of the subject, to some remarks on points which seem to require notice.

As a consequence of preceding reasoning (§ 2), we have

$$\frac{D_x^n}{\Gamma 1+n} x^m = \text{coefficient of } h^n \text{ in } (x+h)^m \ldots\ldots\ (k)$$

for all values of m and n, the development being made according to (a). Now, here we have a meaning attached to D_x^n, which seems a natural extension of the meaning, when n is a positive integer, according to the theory of Lagrange.

But, if true in general, (k) must be true when m is a negative integer. This is the case, which occasioned the principal difficulty in binomial development. It has been seen, that the formula (a) gives a double series, viz. the ordinary series according to increasing powers of h, and that according to decreasing powers of the same letter. I have pointed out, that this result is signalized by the passage through infinity of the general coefficient, when an index is varied continuously from one value to another in the series. This arithmetical failure is undoubted. I have endeavoured to shew how it arises in the case of $(1+x)^{-1}$, which has been the favourite function for illustrating this class of difficulties. I suggest that we have here a case analogous to that of the singular class of series, which are, at once, convergent and divergent. For if we set out, in thought, at one infinity and pass to the other, we may traverse an infinite number of approximative terms, and then an infinite number of divergent ones.

Granting the result as true in form, however unsatisfactory arithmetically, it will be seen, that the formula (k), which constitutes a definition, is still satisfied. The formula holds good when n is a negative integer. The infinite factor $\Gamma 1+n$ disappears by division, leaving

$$D^{-k} x^m = \frac{\Gamma 1+m}{\Gamma 1+m+k} x^{m+k}.$$

10. Let ϕx be any function of x, then

$$\frac{D_x^n}{\Gamma 1+n} \phi x = \text{coefficient of a development of } \phi(x+h) \ldots (k'),$$

according to powers of h to the base under n.

The development will vary with the form, into which ϕx is put. If $\phi(x+h)$ can be developed according to increasing integer powers of $(x+h)$, we can apply (k) to each term. Thus the coefficient of h^n in $\phi(x+h)$ will be of the form

$$\frac{1}{\Gamma 1+n}\left\{\phi 0.\frac{x^{-n}}{\Gamma 1-n}+\phi' 0.\frac{x^{1-n}}{\Gamma 2-n}+\phi'' 0.\frac{x^{2-n}}{\Gamma 3-n}+\ldots\right\}$$

$$=\phi 0.\frac{\sin n\pi}{n\pi}x^{-n}+\phi' 0.\frac{\sin(n-1)\pi}{n.n-1.\pi}$$

$$+\phi'' 0.\frac{\sin(n-2)\pi}{n.n-1.n-2.\pi}x^{2-n}+\ldots\ldots\ldots\ldots(l).$$

When n is a positive integer, every term vanishes up to $\frac{\phi^n 0}{\Gamma n+1}.\frac{\sin(n-n)\pi}{(n-n)\pi}x^{n-n}$ and the form becomes (since $\sin\pm k\pi$ is of the form $\pm 0\pi$)

$$\frac{\phi^n 0}{\Gamma n+1}+\frac{\phi^{n+1}0}{\Gamma n+1}\frac{x}{1}+\frac{\phi^{n+2}0}{\Gamma n+1}\frac{x^2}{1.2}+\ldots,\ i.e.\ \frac{\phi^n x}{\Gamma n+1}.$$

But in order to render the expression (l) complete, we must make an extension analogous to that, which has been given in the case of the exponential function. We have, then,

$$\phi(x+h)=\phi x+\phi' x.h+\phi'' x.\frac{h^2}{1.2}+\ldots+D^{-1}\phi x.\frac{h^{-1}}{\Gamma 0}+D^{-2}\phi x\frac{h^{-2}}{\Gamma -1}+\ldots,$$

and the complete form of (l) is accordingly

$$\phi 0.\frac{\sin n\pi}{n\pi}x^{-n}+\phi' 0.\frac{\sin(n-1)\pi}{n.n-1.\pi}x^{1-n}+\ldots$$

$$+D_0^{-1}\phi 0.\frac{\sin(n+1)\pi}{(n+1)\pi}x^{-1-n}+D_0^{-2}\phi 0.\frac{(n+1)\sin(n+2)\pi}{(n+2)\pi}x^{-2-n}+\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots(l').$$

Since, however, the complementary function proper to D_x^n is of the form

$$k_1x^{-n-1}+k_2x^{-n-2}+\ldots,$$

it is plain, that the portion of the expression newly added may be included in the complementary function.

Developing $\phi(x+h)$ in the form

$$\left\{\phi x.h^0+\phi' x.h+\frac{\phi'' x}{1.2}.\frac{h^2}{1.2}+\ldots\right\}(x+h)^0,$$

we have a form equivalent to (k),

$$x^{-n}\left\{\phi x.\frac{\sin n\pi}{n\pi}+\phi' x\frac{\sin(n-1)\pi}{(n-1)\pi}x+\frac{\phi'' x}{1.2}.\frac{\sin(n-2)\pi}{(n-2)\pi}x^2+\ldots\right\}$$

$$\ldots\ldots\ldots\ldots\ldots(l'),$$

which may be rendered more complete in the manner of (l').

11. To illustrate the difference which may arise by the manner of expressing the function, let us take an example of Prof. Kelland's $\dfrac{(x+h)^2}{x+h-a}$. For the coefficient of h^0 in this function developed to the base index 0, we have

$$(1)\qquad \frac{D_x^{\,0}}{\Gamma 1}\,\frac{x^2}{x-a}=\frac{x^2}{x-a}.$$

But if we write the function as $(x^2+2xh+h^2)(x+h-a)^{-1}$, we have

$$(2)\quad \frac{D_x^{\,0}}{\Gamma 1}.\frac{x^2}{x-a}+2x\frac{D_x^{\,-1}}{\Gamma 0}\,\frac{1}{x-a}+\frac{D_x^{\,-2}}{\Gamma 1}\,\frac{1}{x-a}$$

$$=\frac{x^2}{x-a}+\frac{2x}{\Gamma 0}.\frac{\Gamma 0}{\Gamma 1}+\frac{\Gamma 0}{\Gamma-1.\Gamma 2}(x-a)$$

$$=\frac{x^2}{x-a}+2x-(x-a).$$

"Now we have two expressions of $\dfrac{(x+h)^2}{x+h-a}$ involving a term not containing h; we have of course obtained the sum of them by our process of expanding the term."

The foregoing forms cannot be readily identified by means of the complementary functions. For in (1) there is no such function, and in (2) the constants would have to be infinite. All that the equivalence (k') means is this; if $\phi(x+h)$ can be developed according to powers of $(x+h)$, $\dfrac{D_x^{\,n}}{\Gamma 1+n}\phi x$, ($\phi x$ being similarly developed) will give the corresponding coefficient of h^n in a development to the base index n. The development of $\dfrac{1}{x+h-n}$ gives rise to a double series, of which (2) gives the constant term. We take one part of the double series only, the corresponding constant term is either $\dfrac{x^2}{x-a}$ or $x+a$ according to our choice.

12. I reproduce from a paper by D. F. Gregory (*Cambridge Mathematical Journal*, Vol. II., p. 215), the following remarkable application of the form (l):

If $u = \int dx \int dy \int dz \ldots x^{l-1}.y^{m-1}.z^{n-1}\ldots x+y+z+\ldots \not> 1,$
$$\not< 0,$$

we have, putting v for $x+y+z+\ldots$,

$$u = \int_0^1 dv \int dy \int dz\, y^{m-1}.z^{n-1}.(v-y-z-\ldots)^{l-1}$$

$$= \int_0^1 dv \int dy \int dz \ldots y^{m-1}.z^{n-1}.\epsilon^{-y\frac{d}{dv}}(v-z-\ldots)^{l-1}.$$

Let $\qquad y\dfrac{d}{dv} = t;$ therefore $dy = dt\left(\dfrac{d}{dv}\right)^{-1}.$

Then $\quad u = \int_0^1 dv \int dz\, z^{n-1} \ldots \int dt.t^{m-1}\left(\dfrac{d}{dv}\right)^{-m}(n-z-\ldots)^{l-1}.$

Now $\qquad (v-y)^{l-1} = \epsilon^{-t}.v^{l-1}$ gives $y=0,\ t=0,$
$$y=v,\ t=\infty.$$

Hence $\quad u = \int_0^1 dv \int dz\, z^{n-1}\ldots \int_0^\infty dt.t^{m-1}\epsilon^{-t}\left(\dfrac{d}{dv}\right)^{-m}(v-z\ldots)^{l-1}$

$$= \Gamma m \int_0^1 dv \int dz\, z^{n-1} \ldots \left(\frac{d}{dv}\right)^{-m}(v-z\ldots)^{l-1}.$$

Again, putting $\quad (v-z\ldots)^{l-1} = \epsilon^{-z\frac{d}{dv}}(v-\ldots)^{l-1},$

and assuming $z\dfrac{d}{dv} = s$, we have

$$u = \Gamma m\, \Gamma n \int_0^1 dv \ldots \left(\frac{d}{dv}\right)^{-(m+n)}(v-\ldots)^{l-1}.$$

If then we stop at these variables, we get by (b)

$$u = \frac{\Gamma l\, \Gamma m\, \Gamma n}{\Gamma l+m+n}\int_0^1 dv.v^{l+m+n-1}$$

$$= \frac{\Gamma l\, \Gamma m\, \Gamma n}{\Gamma l+m+n+1}.$$

In the result, however, Γm, Γn are gamma functions in the limited sense, and as to Γl, $\Gamma l+m+n+1$ they also must be limited, on account of the effect of definite integration with regard to v and the limits of t.

(To be continued.)

ON NORMALS TO CONICS AND QUADRICS.

By Frederick Purser, M.A.

Normals to Conics.

If from any point (x_1, y_1) four normals be drawn to the conic $\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0$ (S), their four feet are determined by the intersection of (S) with the rectangular hyperbola whose equation is

$$a^2x_1y-b^2y_1x=(a^2-b^2)xy.$$

Let $A\frac{x}{a}+B\frac{y}{b}+C=0$ (L), $A'\frac{x}{a}+B'\frac{y}{b}+C'=0$ (M)

be the equations of a pair of chords passing through the feet of the normals, then we have identically

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-1-LM-k\{a^2x_1y-b^2y_1x-(a^2-b^2)xy\}=0\ldots(U),$$

whence immediately $AA'=BB'=-CC'=1$. These relations enable us when we are given any condition subsisting amongst the coefficients of the equation of one of the chords (M) to assign at once the corresponding condition in those of the other (L). For we have only to replace A', B', C' in the given condition by $\frac{1}{A}$, $\frac{1}{B}$, $-\frac{1}{C}$. I proceed to make some applications of this principle.

I. Let the chord (M) pass through a fixed point. In this case a linear relation $\lambda A'+\mu B'+\nu C'=0$ holds in the coefficients of (M), hence in the coefficients of (L) we have the relation $\frac{\lambda}{A}+\frac{\mu}{B}-\frac{\nu}{C}=0$. The chord (L) will therefore envelope the parabola $\sqrt{\left(\frac{\lambda x}{a}\right)}+\sqrt{\left(\frac{\mu y}{b}\right)}+\sqrt{(-\nu)}=0$, a parabola touching both axes of (S). If we suppose the fixed point to lie on (S), we obtain the following theorem: *If from any point O on a fixed normal PO to a conic, the three other normals OQ, OR, OS be drawn to the curve, the sides of the triangle QRS will constantly touch a fixed parabola.* It may be shown without difficulty that the

focus of this parabola is the foot of the perpendicular from the centre on the tangent to S at P', the point diametrically opposite to P. It admits also of easy proof that the circle QRS passes through P', and since the focus of the parabola is also a point on this circle, we learn that *if from any point on a fixed normal to a conic there be drawn three other normals to the curve, the circles described through their feet form a system having a common chord of intersection, and this common chord touches the conic.*

II. Let the chord M envelope the conic

$$\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}-1=0 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (\Sigma).$$

In this case we have

$$\frac{\alpha^2}{a^2}A'^2+\frac{\beta^2}{b^2}B'^2=C'^2, \text{ whence } \frac{\alpha^2}{a^2A^2}+\frac{\beta^2}{b^2B^2}=\frac{1}{C^2}.$$

This equation will be found to indicate that the chord (L) is normal to the conic

$$\left(\frac{\alpha^2}{a^4}-\frac{\beta^2}{b^4}\right)^2\left(\frac{a^4x^2}{\alpha^2}+\frac{b^4y^2}{\beta^2}\right)-1=0 \quad\ldots\ldots\ldots (\Sigma').$$

Hence, *If one of the conjugate pair of chords L, M touch a concentric and coaxal conic, the other will be constantly normal to another concentric and coaxal conic, and vice versâ.*

Two particular cases of this theorem may be noticed; 1°. Let (Σ) coincide with the original conic (S). This will evidently occur when the origin of normals lies on the evolute; we infer then that "*if from any point on the evolute of the conic* $\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0$, *the two non-coincident normals be drawn, the chord joining their feet is always normal to the conic whose equation is* $\frac{x^2}{b^2}+\frac{y^2}{a^2}=\frac{a^2b^2}{(a^2-b^2)^2}$, *and therefore envelopes the curve* $\left(\frac{x}{a}\right)^{\frac{2}{3}}+\left(\frac{y}{b}\right)^{\frac{2}{3}}=1$."*

2°. Let (Σ') coincide with (S). Here the origin of the normals lies on (S). For the feet of the four normals from any point O on (S) will be O, Q, R, T. Hence one of the chords L, M will be the normal OQ, the other the chord

* It will be found that this curve is likewise the envelope of the parabolas discussed in (I). It is easy to show geometrically that this should be the case.

RT. This latter chord will then always touch the conic $\frac{x^2}{a^6}+\frac{y^2}{b^6}=\frac{1}{c^4}$. The same conic is of course touched by QR and QT, so that we may state the following theorem: *If from any point on a conic three normals be drawn to it, the sides of the triangle formed by their feet will constantly touch a fixed conic having the same centre and axes as the original.*

This theorem admits of the following generalization: *If the sides of a triangle inscribed in a conic touch a fixed concentric and coaxal conic, the normals to the original conic at the vertices of the triangle meet in a point, and the locus of this point is another concentric and coaxal conic.*

To prove this; referring to the identity (U), from which we started, we obtain for the coordinates (x_1, y_1) of the origin of normals the equations

$$k(a^2-b^2)=\frac{1}{ab}\left(\frac{A}{B}+\frac{B}{A}\right),$$

$$kb^2y_1 \quad =\frac{1}{a}\left(\frac{C}{A}-\frac{A}{C}\right),$$

$$ka^2x_1 \quad =\frac{1}{b}\left(\frac{B}{C}-\frac{C}{B}\right).$$

Suppose now that we are given the equation $\frac{x_1^2}{l^2}+\frac{y_1^2}{m^2}=1$. Then we shall have, writing c^2 for a^2-b^2,

$$c^4\left\{m^2\frac{A^2}{a^2}(B^2-C^2)^2+l^2\frac{B^2}{b^2}(C^2-A^2)^2\right\}-l^2m^2C^2(A^2+B^2)^2=0,$$

an equation which may be written in the form

$$(a^2l^2A^2+b^2m^2B^2-c^4C^2)(A^2B^2c^4-m^2b^2C^2A^2-l^2a^2B^2C^2)$$
$$+A^2B^2C^2(c^8+a^4l^4+b^4m^4-2a^2b^2l^2m^2-2a^2l^2c^4-2b^2m^2c^4)=0.$$

The necessary and sufficient condition that this equation should break up into two factors, is evidently that the coefficient of $(A^2B^2C^2)$ should vanish. This condition may be written in the form $al\pm bm\pm c^2=0$. When this is satisfied, the two factors are, as we have seen,

$$a^2l^2A^2+b^2m^2B^2-C^2 \;\ldots\ldots\ldots\ldots\ldots\ldots\; (H),$$

$$c^4A^2B^2-b^2m^2A^2C^2-l^2a^2B^2C^2 \;\ldots\ldots\ldots\ldots\; (K),$$

A, B, C must then satisfy either $H=0$ or $K=0$. But it is easy to see that when A, B, C satisfies one of these equations,

A', B', C' satisfies the other. Either A, B, C or A', B', C' must then satisfy $H=0$, the interpretation of which result is, that one of each pair of chords which pass through the four feet of normals from (x_1, y_1), touches the conic $\frac{x^2}{l'^2}+\frac{y^2}{m'^2}=1$, where $l'=\frac{a^2l}{c^2}$, $m'=\frac{b^2m}{c^2}$. But by virtue of the condition $al \pm bm \pm c^2 = 0$, we have $\frac{l'}{a} \pm \frac{m'}{b} \pm 1 = 0$, which is the condition that a triangle may be at once inscribed in (a, b) and circumscribed to (l', m'). It will then follow, denoting the four feet of normals by P, Q, R, S, that PQ, PR, QR touch the conic (l', m'), while PS, QS, RS, touch a certain curve of the fourth class.

Normals to Quadrics.

III. From any point x_1, y_1, z_1 six normals can be drawn to the quadric $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1=0$ (S). Their feet will satisfy besides the equation to the quadric, the three equations

$$(b^2-c^2)\,yz + c^2z_1y - b^2y_1z = 0 \quad \ldots\ldots\ldots\ldots\ldots (U),$$

$$(c^2-a^2)\,zx + a^2x_1z - c^2z_1x = 0 \quad \ldots\ldots\ldots\ldots\ldots (V),$$

$$(a^2-b^2)\,xy + b^2y_1x - a^2x_1y = 0 \quad \ldots\ldots\ldots\ldots (W).$$

Now if

$$A\frac{x}{a}+B\frac{y}{b}+C\frac{z}{c}+D=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (L),$$

$$A'\frac{x}{a}+B'\frac{y}{b}+C'\frac{z}{c}+D'=0 \quad \ldots\ldots\ldots\ldots\ldots (M),$$

be two planes passing through the six feet of the normals, we must have identically

$$S-LM=\lambda U+\mu V+\nu W.$$

From this identity arise the equations

$$AA'=BB'=CC'=-DD'=1 \quad \ldots\ldots\ldots\ldots (1),$$

$$\left.\begin{aligned} \lambda(b^2-c^2) &= \frac{B^2-C^2}{BCbc} \\ \mu(c^2-a^2) &= \frac{C^2-A^2}{CAca} \\ \nu(a^2-b^2) &= \frac{A^2-B^2}{ABab} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (2),$$

$$\left.\begin{aligned} \nu b^2 y_1 - \mu c^2 z_1 &= \frac{1}{a^2} \cdot \frac{D^2 - A^2}{AD} \\ \lambda c^2 z_1 - \nu a^2 x_1 &= \frac{1}{b^2} \cdot \frac{D^2 - B^2}{BD} \\ \mu a^2 x_1 - \lambda b^2 y_1 &= \frac{1}{c^2} \cdot \frac{D^2 - C^2}{CD} \end{aligned}\right\} \ldots\ldots\ldots\ldots (3).$$

From the system (3), we deduce

$$\frac{\lambda}{aA}(A^2 - D^2) + \frac{\mu}{bB}(B^2 - D^2) + \frac{\nu}{cC}(C^2 - D^2) = 0,$$

whence, by substituting for λ, μ, ν from (2), we obtain the following relations in the coefficients of L,

$$\frac{B^2 + C^2}{b^2 - c^2}(A^2 - D^2) + \frac{(C^2 + A^2)}{c^2 - a^2}(B^2 - D^2)$$
$$+ \frac{A^2 + B^2}{a^2 - b^2}(C^2 - D^2) = 0 \ldots\ldots\ldots (4),$$

or, as it may be otherwise written,

$$(b^2 - c^2)^2 (B^2C^2 - A^2D^2) + (C^2 - a^2)^2 (C^2A^2 - B^2D^2)$$
$$+ (a^2 - b^2)^2 (A^2B^2 - C^2D^2) = 0 \ldots\ldots\ldots\ldots (5).$$

This relation involving only the coefficients A, B, C, D leads to the following interesting theorem, which is, so far as I am aware, new: *If from any point three normals be drawn to a quadric, the plane passing through their feet will always touch a fixed surface.* This theorem may be likewise stated as a porism, thus: *The problem to find three points in a given plane section of a quadric, such that the corresponding normals meet in a point is either indeterminate or impossible.*

In this latter form, the truth of the theorem may be independently established as follows: Let (x', y', z'), (x'', y'', z'') be two points on the quadric whose normals intersect. Then it is easily seen that we have

$$a^2 (x' - x'') (y'z'' - y''z') + b^2 (y' - y'') (z'x'' - z''x')$$
$$+ c^2 (z' - z'') (x'y'' - x''y') = 0 \ldots\ldots (6).*$$

Suppose now that (x', y', z'), (x'', y'', z'') lie on the plane $x \cos\alpha + y \cos\beta + z \cos\gamma - p = 0$. It will be found convenient

* From this condition it is at once apparent, that if normals at the points of intersection of a line with a quadric meet, then normals at its points of intersection with any similar or any confocal quadric likewise meet.

to transform to a system of trilinear coordinates in this plane, the triangle of reference being formed by the lines of intersection of the plane with the three principal planes. If we denote these new coordinates by ξ, η, ζ, we shall have the formulæ of transformation $x = \xi \sin\alpha$, $y = \eta \sin\beta$, $z = \zeta \sin\gamma$, whence the equation of the curve of section will become

$$\frac{\xi^2}{a^2}\sin^2\alpha + \frac{\eta^2}{b^2}\sin^2\beta + \frac{\zeta^2}{c^2}\sin^2\gamma$$
$$= \frac{1}{p^2}(\xi\sin\alpha\cos\alpha + \eta\sin\beta\cos\beta + \zeta\sin\gamma\cos\gamma)^2.$$

Similarly, transforming equation (6), we find for the condition that the normals to the quadric at $\xi_1, \eta_1, \zeta_1, \xi_2, \eta_2, \zeta_2$ should intersect

$$a^2(\xi_1 - \xi_2)(\eta_1\zeta_2 - \eta_2\zeta_1) + b^2(\eta_1 - \eta_2)(\zeta_1\xi_2 - \xi_1\zeta_2)$$
$$+ c^2(\zeta_1 - \zeta_2)(\xi_1\eta_2 - \xi_2\eta_1) = 0.$$

Now we have evidently

$$(\xi_1 - \xi_2)\sin\alpha\cos\alpha + (\eta_1 - \eta_2)\sin\beta\cos\beta + (\zeta_1 - \zeta_2)\sin\gamma\cos\gamma = 0,$$

a relation which enables us to write the condition for the intersection of normals in the form

$$\frac{(b^2 - c^2)\sin\alpha\cos\alpha}{\eta_1\zeta_2 - \eta_2\zeta_1} + \frac{(c^2 - a^2)\sin\beta\cos\beta}{\zeta_1\xi_2 - \xi_1\zeta_2} + \frac{(a^2 - b^2)\sin\gamma\cos\gamma}{\xi_1\eta_2 - \xi_2\eta_1},$$

or

$$\frac{(b^2 - c^2)\sin\alpha\cos\alpha}{l} + \frac{(c^2 - a^2)\sin\beta\cos\beta}{m} + \frac{(a^2 - b^2)\sin\gamma\cos\gamma}{n} = 0,$$

where $l\xi + m\eta + n\zeta = 0$ is the equation of the line joining ξ_1, η_1, ζ_1 to ξ_2, η_2, ζ_2. This equation indicates that this line touches the curve, whose equation is

$$\sqrt{\{(b^2 - c^2)\xi\sin\alpha\cos\alpha\}} + \sqrt{\{(c^2 - a^2)\eta\sin\beta\cos\beta\}}$$
$$+ \sqrt{(a^2 - b^2)\zeta\sin\gamma\cos\gamma} = 0 \;\ldots\ldots\; (7),$$

and which is therefore a parabola inscribed in the triangle of reference. Now, if normals to the quadric at three points on the curve of section meet in a point, this parabola must be such that a triangle can be at once circumscribed to it, and inscribed in the curve of section. Hence, in order that it should be possible to find three such points on the curve of section, a certain condition must hold in the coefficients of the equation of the plane of section. Conversely, if this condition hold, an indefinite number of systems of points of the required kind may be found. For there are, in this case, an indefinite number of triangles inscribed in the curve of section, and

circumscribed to the parabola (7), and, from what we have previously proved, it follows that normals to the quadric at the vertices of any such triangle meet in pairs. Now it is evident, geometrically, that normals to a quadric at three points, cannot, in general, meet in pairs except they meet in one point. Hence the problem in question is porismatic, as was to be proved. It may be well to verify that the condition to which we are led by this method is identical, as it should be, with that found by the former. The condition is (see Salmon, *Lessons on Higher Algebra*, p. 107)

$$\Theta'^2 = 4\Theta\Delta',$$

where Δ, Θ, Θ', Δ' are the fundamental invariants of the parabola and the conic of section. On calculation Θ' will be found to vanish identically, and the condition reduces to $\Theta = 0$, or

$$\begin{aligned}p^2\{a^2\cos^2\alpha\,(b^2-c^2)^2+b^2\cos^2\beta\,(c^2-a^2)^2+c^2\cos^2\gamma\,(a^2-b^2)^2\}&\\ =b^2c^2\,(b^2-c^2)^2\cos^2\beta\cos^2\gamma+c^2a^2\,(c^2-a^2)^2\cos^2\gamma\cos^2\alpha&\\ +a^2b^2\,(a^2-b^2)^2\cos^2\alpha\cos^2\beta.&\end{aligned}$$

This condition is readily seen to be identical with (5).

Let us now revert to the equations (3) of the original system. We saw that it was possible by the aid of equations (2) to eliminate from them λ, μ, ν, x_1, y_1, z_1, thus obtaining a condition amongst the coefficients of the equation of the plane alone. When this condition is satisfied, equations (3) become equivalent to two linear equations in x_1, y_1, z_1, showing that when the plane (L) is given, this point moves along a certain given line. From what has already been proved, it will then follow that if from any point on this line six normals be drawn to the quadric, three of their feet will lie on the plane (L) and three on the plane (M). Consequently the line meets all the normals drawn at points on either curve of section. It is also evident that the line meets both curves of section, for if O be either of the points where it meets the quadric, O will be also one of the corresponding feet of normals, and will therefore lie on one of the curves of section. The following construction may be given for the line when any point K on it is known. Let P, Q, R, S, T, U be the feet of the six normals from K, and let the plane P, Q, R be denoted by (L), S, T, U by (M). From K drop a perpendicular KV on the plane (L). Then PV, QV, RV are evidently normals to the curve of section of (L). Let N be the fourth point on this curve whose normal passes through V, and let VN meet the curve again in H. Similarly, let H' be

determined in the curve of section of (M), then the line HH' will pass through K, and will be the required line for the system (L), (M). As there can be described ten distinct pairs of planes passing through the six feet of normals from K, we learn that ten lines of this kind (which we may call, for convenience, locus lines) can be drawn through any point.

Let us now consider the points in which the locus line corresponding to the planes (L), (M) meets the surface of centres. Let O be such a point, then the six normals from O will be OA, OA, OB, OC, OD, OE. Now, viewing the distribution of the feet of the normals between the two sections (L), (M) as it affects the two coincident normals OA, OA, we distinguish three possible cases: (1°) (L) contains OA, OA; (2°) (M) contains OA, OA; (3°) (L) contains OA and (M) contains OA. The two first cases correspond to the points on the curves of section of (L), (M), where they touch lines of curvature. Now these are clearly the points of contact with either curve of section of common tangents to it and the parabola (7). Cases (1°) and (2°) correspond then to eight points of intersection of the locus line with the surface of centres. The number of intersections being twelve in all, four remain to be accounted for by (3°), and correspond therefore to the *two* points of intersection of the curves of section (L), (M). Each of these latter must then answer to two coincident intersections of the locus line with the surface of centres. Hence the important result, "all locus lines are double tangents to the surface of centres." The following independent geometric proof of this proposition has been suggested to me. Let Q be a point on the curve of section of one of the planes (L) very near a point where the two curves meet, P, R two points on the other curve of section lying on the two lines of curvature through Q. Now the locus line, as we have proved, meets the normals at P, Q, R. But in the limit, the normals at P, Q are co-planar meeting in one centre of curvature, and those at Q, R co-planar meeting in the other; hence the locus line must pass through one of the centres of curvature, and lie in the plane of the two normals which meet in the other. In other words, it passes through a point on the surface of centres, and lies in the corresponding tangent plane. The locus line is therefore a tangent to the surface of centres at the two points corresponding to the points of intersection of the curves of section of (L) and (M).

Lota, Blackrock, Co. Dublin,
April, 1866.

SUPPLEMENTARY NOTE ON NASIK CUBES.

By Andrew Frost, M.A.

Arrangement of 7 cubes, each containing 7^3 or 343 cells, so that each separate cube shall possess nasical properties and the cells be numbered from 1 to 7^4, or 1 to 2401.

In the investigation of the knight's move in different directions from the same square, it appeared that there were 4 paths which it might take without passing over the same square, as when there are 7 cells in a side. As only 3 of these paths were required in the construction of a single nasical cube, in using an analogous method with the 4 paths, we are enabled to construct 7 such cubes, each containing 7^3 cells, on which 7^4 cells may be written the numbers from 1 to 7^4 or 2401, so that each of the 7 cubes shall possess nasical properties.

Let π, ρ, σ, τ, (fig. 8) be placed in the corner cell of one of the 7 cubes, and let each letter be dispersed along the 4 different knight's paths in the plane *yox*, *zoz*, and let each letter which occurs in the plane *xoy* pass over a similar path in the plane parallel to *xoz* in which it lies, the knight's path in the parallel planes being in similar directions, we shall find that each letter has arranged itself in each plane perpendicular to *ox* in the knight's path. If we write π_2, $\pi_3 \ldots \pi_7$ after π_1 in the consecutive cells parallel to *ox* throughout the cube, and similarly for ρ, σ, τ; the sum of the 7 figures agreeably with the nasical additions throughout the cube, will be

$$\Sigma\pi + \Sigma\rho + \Sigma\sigma + \Sigma\tau.$$

If we take the second cube and throughout it in the place of π_1, $\pi_2 \ldots \pi_7$ write π_2, $\pi_3 \ldots \pi_7$, π_1, we shall obtain different combinations of the letters in each cell, with the same sum for the additions of 7 cells.

If we continue this process in filling up the cells of the remaining 7 cubes, and write for $\pi_1 \ldots \pi_7$, $\rho_1 \ldots \rho_7$, $\sigma_1 \ldots \sigma_7$, the Nos. 0, 1, 2…6 for $\tau_1 \ldots \tau_4$, the Nos. 1, 2…7 in any order, we shall have the cells of the 7 cubes containing the consecutive Nos. from 1 to 7^4 in the 7^y scale, which can be reduced to the denary, thus giving the Nos. from 1 to 2401.

ON METACENTRE IN A LIQUID OF VARIABLE DENSITY.

By W. H. Besant, M.A., St. John's College, Cambridge.

I HAVE investigated, by several different methods, a formula for the determination of the metacentre, in the case of a heavy body floating in a liquid of variable density, and the present note contains two of these methods. For the sake of simplicity I give only the case in which the body is symmetrical with regard to the vertical plane of displacement, and also with regard to a perpendicular plane.

We can imagine a liquid of variable density constructed by successive elementary accretions of density, these accretions commencing at successive levels, which descend at elementary intervals.

Let V be the volume displaced by a body immersed in such a liquid, A_0 the surface, $A_1, A_2, \ldots$ a series of horizontal sections, $v_1, v_2, \ldots$ the volumes, and $\rho_1, \rho_2, \rho_3, \ldots$ the densities of the successive strata.

Then the whole mass displaced

$$
\begin{aligned}
&= v_1\rho_1 + v_2\rho_2 + v_3\rho_3 + \ldots \\
&= v_1\rho_1 + v_2(\rho_1 + \delta\rho_1) + v_3(\rho_1 + \delta\rho_1 + \delta\rho_2) + \ldots \\
&= V\rho_1 + (V - v_1)\,\delta\rho_1 + (V - v_1 - v_2)\,\delta\rho_2 + \ldots,
\end{aligned}
$$

and is thus made up of a series of liquids having $A_0, A_1, A_2, \ldots$ for their successive surfaces.

If a heavy body, floating in homogeneous liquid, be slightly displaced in a vertical plane through a small angle θ, the resultant fluid action then consists of a single vertical force $g\rho V$ acting through the centre of gravity (H) of the liquid displaced, and of a couple $g\rho V.HM.\theta$ or $g\rho k^2 A.\theta$.

In the present case, let U be the mass of the liquid displaced, H its centre of gravity when the body is in equilibrium, and M the metacentre.

In the position of displacement, the resultant action will consist of a series of vertical forces acting through the respective centres of gravity of the bodies V, $V - v_1$, $V - v_1 - v_2$, &c., and of a series of couples, each of the form $g\rho.k^2A.\theta$.

The vertical forces will be equivalent to a single force gU, acting through H, and the series of couples will be equivalent to the couple $gU.HM.\theta$.

Hence, taking ρ_0 for the density at the surface, and ρ the density at a depth CN,

$$U.HN = \rho_0 k_0^2 A_0 + \Sigma\,(k^2 A\delta\rho),\ \text{(fig. 9)},$$

k^2A being the moment of inertia of the area of the section PNQ about the horizontal line through its centre of gravity perpendicular to the plane of displacement, and $k_0^2A_0$ the equivalent expression for the surface ACB. But

$$\begin{aligned}\Sigma\,(k^2A\delta\rho) &= \int k^2 A d\rho \\ &= (\rho k^2 A)_C^O - \int \rho d\,(k^2A);\end{aligned}$$

and, if vertical lines through the boundaries of the sections PQ, $P'Q'$ meet the surface-plane ACB in curves Ll, $L'l'$, $-d\,(k^2A)$ is the moment of inertia about the horizontal line through C of the portion of ACB between those curves.

Hence $$U.HM = \iint \rho x^2 dy\, dx,$$

CA being the axis of x, and ρ the density at the extremity of the vertical ordinate of the body drawn through the point (x, y). If the body have a flat base, ρ will be constant for the corresponding portion of the integral.

The above formula may also be obtained in the following manner:

If a body, floating between two liquids, be slightly displaced, the couple which is thereby called into action is

$$g\rho\theta k^2 A - g\rho'\theta k^2 A,$$

ρ, ρ' being the density of the lower and upper liquids respectively, and k^2A the moment of inertia of the common section.

Hence, if, in fig. 9, $\delta\rho$ be the increase of density in passing across the section PQ, the couple for that section is

$$g\delta\rho\,.\,\theta\,.\,k^2A.$$

Therefore $$U.HM = \rho_0 k_0^2 A_0 + \int k^2 A d\rho,$$

and the rest of the process is the same as before.

January, 1866.

ON A LOCUS DERIVED FROM TWO CONICS.

By Professor A. CAYLEY.

REQUIRED the locus of a point which is such that the pencil formed by the tangents through it to two given conics has a given anharmonic ratio.

Suppose, for a moment, that the equation of the tangents to the first conic is $(x-ay)(x-by)=0$, and that of the tangents to the second conic is $(x-cy)(x-dy)=0$, and write

$$A=(a-b)(c-d),$$
$$B=(a-c)(d-b),$$
$$C=(a-d)(b-c),$$

so that
$$A+B+C=0,$$

write also
$$k_1=\frac{B}{A},\quad k_2=\frac{C}{A},$$

then the anharmonic ratio of the pencil will have a given value k if

$$(k-k_1)(k-k_2)=0;$$

that is, if
$$k^2+k+\frac{BC}{A^2}=0,$$

or, what is the same thing, if

$$A^2(2k+1)^2+4BC-A^2=0;$$

that is, if
$$A^2(2k+1)^2-(B-C)^2=0,$$

where
$$A^2=(a-b)^2(c-d)^2,$$
$$B-C=(a+b)(c+d)-2(ab+cd),$$

are each of them symmetrical in regard to a b, and in regard to c, d respectively.

Let the equations of the two conics be

$$U=(a,\ b,\ c,\ f,\ g,\ h\between x,\ y,\ z)^2=0,$$
$$U'=(a',\ b',\ c',\ f',\ g',\ h'\between x,\ y,\ z)^2=0,$$

and let (α, β, γ) be the coordinates of the variable point. Putting as usual

$$(A, B, C, F, G, H)$$
$$= (bc-f^2, ca-g^2, ab-h^2, gh-af, hf-bg, fg-ch),$$
$$K = abc - af^2 - bg^2 - ch^2 + 2fgh,$$

the equation of the tangents to the first conic is

$$(A, B, C, F, G, H \between X, Y, Z)^2 = 0,$$

where $\quad X = \gamma y - \beta z, \quad Y = \alpha z - \gamma x, \quad Z = \beta x - \alpha y,$

and therefore $\quad \alpha X + \beta Y + \gamma Z = 0.$

Hence substituting for Z the value $-\frac{1}{\gamma}(\alpha X + \beta Y)$, we find, for the equation of the tangents, an equation of the form $\mathrm{a}X^2 + 2\mathrm{h}XY + \mathrm{b}Y^2 = 0$, which has, in effect, been taken to be $(X - aY)(X - bY) = 0$; that is, we have

$$1 : a+b : ab = \mathrm{a} : -2\mathrm{h} : \mathrm{b}.$$

And, in like manner, if the accented letters refer to the second conic

$$1 : c+d : cd = \mathrm{a}' : -2\mathrm{h}' : \mathrm{b}'.$$

Substituting for a, h, b their values, and for a′, h′, b′ the corresponding values, we find

$1 : a+b : ab$	$1 : c+d : cd$
$= A\gamma^2 - 2G\gamma\alpha + C\alpha^2$	$= A'\gamma^2 - 2G'\gamma\alpha + C'\alpha^2$
$: -2(H\gamma^2 - F\alpha\gamma - G\beta\gamma + C\alpha\beta)$	$: -2(H'\gamma^2 - F'\alpha\gamma - G'\beta\gamma + C'\alpha\beta)$
$: B\gamma^2 - 2F\beta\gamma + C\beta^2.$	$: B'\gamma^2 - 2F'\beta\gamma + C'\beta^2.$

We then have

$$(a-b)^2 = (a+b)^2 - 4ab$$
$$= 4(H\gamma^2 - F\alpha\gamma - G\beta\gamma + C\alpha\beta)^2$$
$$-4(A\gamma^2 - 2G\gamma\alpha + C\alpha^2)(B\gamma^2 - 2F\beta\gamma + C\beta^2)$$
$$= -4\gamma^2(BC - F^2, \ldots \between \alpha, \beta, \gamma)^2$$
$$= -4\gamma^2 K(a, \ldots \quad \between \alpha, \beta, \gamma)^2,$$

and similarly

$$(c-d)^2 = -4\gamma^2 K'(a', \ldots \between \alpha, \beta, \gamma)^2.$$

We have, moreover,

$$(a+b)(c+d)-2(ab+cd)$$

$$\begin{aligned}
= \;& 4(H\gamma^2 - F\alpha\gamma - G\beta\gamma + C\alpha\beta)(H'\gamma^2 - F'\alpha\gamma - G'\beta\gamma + C'\alpha\beta) \\
& -2(B\gamma^2 - 2F\beta\gamma + C\beta^2)(A'\gamma^2 - 2G'\gamma\alpha + C'\alpha^2) \\
& -2(B'\gamma^2 - 2F'\beta\gamma + C'\beta^2)(A\gamma^2 - 2G\gamma\alpha + C\alpha^2) \\
= & -2\gamma^2(BC' + B'C - 2FF', \ldots \between \alpha, \beta, \gamma)^2,
\end{aligned}$$

and substituting the foregoing values, we find

$$4(2k+1)^2 KK'(a, \ldots \between \alpha, \beta, \gamma)^2 (a', \ldots \between \alpha, \beta, \gamma)^2 - \{(BC' + B'C - 2FF', \ldots \between \alpha, \beta, \gamma)^2\}^2 = 0,$$

or putting for shortness

$$\Theta = (BC' + B'C - 2FF', ., ., GH' + G'H - AF' - A'F, ., . \between \alpha, \beta, \gamma)^2,$$

the equation of the locus is

$$4(2k+1)^2 KK'.UU' - \Theta^2 = 0,$$

where (α, β, γ) are current coordinates. The locus is thus a quartic curve having quadruple contact with each of the conics $U=0$, $U'=0$; viz. it touches them at their points of intersection with the conic $\Theta=0$, *which is the locus of the point such that the four tangents form a harmonic pencil.*

The equation may be written somewhat more elegantly under the form

$$4(2k+1)^2.KU.K'U' - \Theta^2 = 0;$$

viz. in this equation we have

$$\begin{aligned}
KU &= (BC - F^2, \ldots \between \alpha, \beta, \gamma)^2, \\
K'U' &= (B'C' - F'^2, \ldots \between \alpha, \beta, \gamma)^2, \\
\Theta &= (BC' + B'C - 2FF', \ldots \between \alpha, \beta, \gamma)^2.
\end{aligned}$$

In the last form the equation is expressed in terms of the coefficients $(A, \ldots)$, $(A', \ldots)$ of the *line* equations of the conics, viz. these may be taken to be

$$(A, \ldots \between \xi, \eta, \zeta)^2 = 0, \quad (A', \ldots \between \xi, \eta, \zeta)^2 = 0.$$

In particular, if each of the conics break up into a pair of points, viz. (l, m, n) and (p, q, r) for the first conic,

(l', m', n') and (p', q', r') for the second conic, then the line equations are

$$2(l\xi + m\eta + n\zeta)(p\xi + q\eta + r\zeta) = 0,$$
$$2(l'\xi + m'\eta + n'\zeta)(p'\xi + q'\eta + r'\zeta) = 0,$$

so that
$$A = 2lp, \;..\; F = mr + nq, \;..$$
$$A' = 2l'p', \;..\; F' = m'r' + n'q', \;..$$
$$(BC - F^2, \ldots) = -(mr - nq,\; np - lr,\; lq - mp)^2,$$
$$(B'C' - F'^2, \ldots) = -(m'r' - n'q',\; n'p' - l'r',\; l'q' - m'p')^2,$$
$$BC' + B'C - 2FF'$$
$$= 2\{(mn' - m'n)(qr' - q'r) - (mr' - nq')(m'r - n'q), \ldots\},$$

and substituting these values the equation is

$$(2k+1)^2 \begin{vmatrix} \alpha, & \beta, & \gamma \\ l, & m, & n \\ p, & q, & r \end{vmatrix}^2 \begin{vmatrix} \alpha, & \beta, & \gamma \\ l', & m', & n' \\ p', & q', & r' \end{vmatrix}^2$$
$$-\left\{\begin{vmatrix} \alpha, & \beta, & \gamma \\ l, & m, & n \\ l', & m', & n' \end{vmatrix}\begin{vmatrix} \alpha, & \beta, & \gamma \\ p, & q, & r \\ p', & q', & r' \end{vmatrix} - \begin{vmatrix} \alpha, & \beta, & \gamma \\ l, & m, & n \\ p', & q', & r' \end{vmatrix}\begin{vmatrix} \alpha, & \beta, & \gamma \\ l', & m', & n' \\ p, & q, & r \end{vmatrix}\right\}^2 = 0,$$

which if A, B, C denote

$$\begin{vmatrix} \alpha, & \beta, & \gamma \\ l, & m, & n \\ p, & q, & r \end{vmatrix}\begin{vmatrix} \alpha, & \beta, & \gamma \\ l', & m', & n' \\ p', & q', & r' \end{vmatrix}, \quad \begin{vmatrix} \alpha, & \beta, & \gamma \\ l, & m, & n \\ l', & m', & n' \end{vmatrix}\begin{vmatrix} \alpha, & \beta, & \gamma \\ p', & q', & r' \\ p, & q, & r \end{vmatrix},$$
$$\begin{vmatrix} \alpha, & \beta, & \gamma \\ l, & m, & n \\ p', & q', & r' \end{vmatrix}\begin{vmatrix} \alpha, & \beta, & \gamma \\ l', & m', & n' \\ p, & q, & r \end{vmatrix}.$$

respectively, $(A + B + C = 0)$ is, in fact, the equation

$$(2k+1)^2 A^2 - (B - C)^2 = 0,$$

or, what is the same thing,

$$\left(k - \frac{B}{A}\right)\left(k - \frac{C}{A}\right) = 0,$$

that is
$$k = \frac{B}{A} \text{ or } k = \frac{C}{A},$$

either of which expresses the anharmonic property of the points of a conic in the form given by the theorem *ad quatuor lineas.*

Reverting to the case of two conics, then if these be referred to a set of conjugate axes, the equations will be

$$ax^2 + by^2 + cz^2 = 0,$$
$$a'x^2 + b'y^2 + c'z^2 = 0,$$

we have $K = abc$, $K' = a'b'c'$,

$$\Theta = (bc' + b'c)\,aa'x^2 + (ca' + c'a)\,bb'y^2 + (ab' + a'b)\,cc'z^2,$$

and the equation of the quartic curve is

$$4\,(2k+1)^2\,abca'b'c'\,(ax^2 + by^2 + cz^2)\,(a'x^2 + b'y^2 + c'z^2)$$
$$- \{(bc' + b'c)\,aa'x^2 + (ca' + c'a)\,bb'y^2 + (ab' + a'b)\,cc'z^2\}^2 = 0.$$

I suppose in particular that the two conics are

$$x^2 + my^2 - 1 = 0,$$
$$mx^2 + \ \ y^2 - 1 = 0,$$

the equation of the quartic is

$$4(2k+1)^2m^2(x^2+my^2-1)(mx^2+y^2-1)-\{(m^2+m)(x^2+y^2)-m^2-1\}^2=0.$$

Or putting $\lambda = \dfrac{(m+1)^2}{4\,(2k+1)^2}$, this is

$$\lambda\left(x^2 + y^2 - \frac{m^2+1}{m^2+m}\right)^2 - (x^2 + my^2 - 1)\,(mx^2 + y^2 - 1) = 0.$$

To fix the ideas, suppose that m is positive and > 1, so that each of the conics is an ellipse, the major semi-axis being $= 1$, and the minor semi-axis being $= \dfrac{1}{\sqrt{(m)}}$. For any real value of k the coefficient λ is positive, and it may accordingly be assumed that λ is positive.

We have $\dfrac{m^2+1}{m\,(m+1)} > \dfrac{1}{m} < 1$, or the radius of the circle is intermediate between the semi-axes of the ellipses, hence the points of contact on each ellipse are real points.

Writing for shortness

$$\alpha = \frac{m^2+1}{m^2+m},$$

the equation is

$$(x^2 + my^2 - 1)\,(mx^2 + y^2 - 1) - \lambda\,(x^2 + y^2 - \alpha)^2 = 0.$$

And for the points on the axis of x, we have

$$(x^2 - 1)\,(mx^2 - 1) - \lambda\,(x^2 - \alpha)^2 = 0,$$

that is $(m-\lambda)x^4+\{-(1+m)+2\lambda\alpha\}x^2+(1-\lambda\alpha^2)=0$,
and thence

$(m-\lambda)x^2=\frac{1}{2}(1+m)-\lambda\alpha\pm\frac{1}{2}\sqrt{\{(m-1)^2+4\lambda(1-\alpha)(1-m\alpha)\}}$,
or, substituting for α its value, this is

$$(m-\lambda)x^2=\tfrac{1}{2}(m+1)-\frac{\lambda\left(m+\frac{1}{m}\right)}{m+1}\pm\frac{\frac{1}{2}(m-1)}{m+1}\sqrt{\{(m+1)^2-4\lambda\}}.$$

Remarking that the values $\dfrac{(m+1)^2}{\left(m+\frac{1}{m}\right)^2}$, m, $\dfrac{(m+1)^2}{4}$ are in the order of increasing magnitude;

Consider the case $\lambda=\dfrac{1}{\alpha^2}$, $=\dfrac{(m+1)^2}{\left(m+\frac{1}{m}\right)^2}$, we have

$$(m-\lambda)x^2=\tfrac{1}{2}(m+1)-\frac{m+1}{m+\frac{1}{m}}\pm\frac{\frac{1}{2}(m-1)\left(m-\frac{1}{m}\right)}{\left(m+\frac{1}{m}\right)}$$

$$=\frac{(m+1)\frac{1}{2}\left(m+\frac{1}{m}-2\right)\pm\frac{1}{2}(m-1)\left(m-\frac{1}{m}\right)}{\left(m+\frac{1}{m}\right)}.$$

Or observing that

$$(m+1)\left(m+\frac{1}{m}-2\right)=(m+1)\frac{1}{m}(m-1)^2$$

$$=\frac{1}{m}(m-1)(m^2-1)=(m-1)\left(m-\frac{1}{m}\right),$$

this is $\qquad (m-\lambda)x^2=0$, or $\dfrac{(m-1)\left(m-\frac{1}{m}\right)}{m+\frac{1}{m}}$,

or, what is that same thing,

$$\frac{(m-1)(m^3+2m^2-1)}{m\left(m+\frac{1}{m}\right)^2}x^2=0,\ \text{or}\ \frac{(m-1)\left(m-\frac{1}{m}\right)}{m+\frac{1}{m}},$$

$$x^2=0,\ \text{or}\ \frac{\left(m^2-\frac{1}{m^2}\right)m}{m^3+2m^2-1}.$$

The next critical value is $\lambda = m$. The curve here is

$$(x^2 + my^2 - 1)(mx^2 + y^2 - 1) - m(x^2 + y^2 - \alpha)^2 = 0,$$

that is $m(x^4 + y^4) + (1 + m^2)x^2y^2 - (m+1)(x^2 + y^2) + 1$

$$- m(x^4 + y^4) - \quad 2m \quad x^2y^2 + \quad 2m\alpha \quad (x^2 + y^2) - m\alpha^2 = 0,$$

that is

$$(m-1)^2x^2y^2 + (2m\alpha - m - 1)(x^2 + y^2) + 1 - m\alpha^2 = 0,$$

or, substituting for α its value,

$$2m\alpha - m - 1 = \frac{2m^2 + 2}{m+1} - (m+1) = \frac{(m-1)^2}{m+1},$$

$$1 - m\alpha^2 = 1 - \frac{(m^2+1)^2}{m(m+1)^2} = -\frac{(m-1)^2(m^2 + m + 1)}{m(m+1)^2};$$

the equation is

$$x^2y^2 + \frac{1}{m+1}(x^2 + y^2) - \frac{m^2 + m + 1}{m(m+1)^2} = 0,$$

or, as this may also be written,

$$\left(x^2 + \frac{1}{m+1}\right)\left(y^2 + \frac{1}{m+1}\right) - \frac{1}{m} = 0,$$

which has a pair of imaginary asymptotes parallel to the axis of x, and a like pair parallel to the axis of y, or what is the same thing, the curve has two isolated points at infinity, one on each axis.

The next critical value is $\lambda = \frac{(m+1)^2}{4}$; the curve here reduces itself to the four lines

$$\left\{(x+y)^2 - \frac{m+1}{m}\right\}\left\{(x-y)^2 - \frac{m+1}{m}\right\} = 0;$$

and it is to be observed that when λ exceeds this value, or say $\lambda > \frac{(m+1)^2}{4}$, the curve has no real point on either axis; but when $\lambda = \infty$, the curve reduces itself to $(x^2 + y^2 - \alpha)^2 = 0$, *i.e.* to the circle $x^2 + y^2 - \alpha = 0$ twice repeated, having in this special case real points on the two axes.

It is now easy to trace the curve for the different values of λ. The curve lies in every case within the unshaded regions of the figure (fig. 10), (except in the limiting cases after-mentioned); and it also touches the two ellipses and the

four lines at the eight points k, at which points it also cuts the circle; but it does not cut or touch the four lines, the two ellipses, or the circle, except at the points k. Considering λ as varying by successive steps from 0 to ∞;

$\lambda = 0$, the curve is the two ellipses.

$\lambda < \frac{(m+1)^2}{\left(m+\frac{1}{m}\right)^2}$, the curve consists of two ovals, an exterior sinuous oval lying in the four regions a and the four regions b; and an interior oval lying in the region c.

$\lambda = \frac{(m+1)^2}{\left(m+\frac{1}{m}\right)^2}$, there is still a sinuous oval as above, but the interior oval has dwindled to a conjugate point at the centre.

$\lambda > \frac{(m+1)^2}{\left(m+\frac{1}{m}\right)^2} < m$; $\lambda = m$; $\lambda > m < \frac{(m+1)^2}{4}$; there is no interior oval, but only a sinuous oval as above; which as λ increases approaches continually nearer to the four sides of the square. For the critical value $\lambda = m$, there is no change in the general form, but the curve has for this value of λ, two conjugate points, one on each axis at infinity.

$\lambda = \frac{(m+1)^2}{4}$, the curve becomes the four lines

$\lambda > \frac{(m+1)^2}{4}$, the curve lies wholly in the four regions a and the four regions e, consisting thereof of four detached sinuous ovals. As λ deviates less from the value $\frac{(m+1)^2}{4}$, each oval approaches more nearly to the infinite trilateral formed by the side and infinite line-portions which bound the regions d, e to which the oval belongs. And as λ departs from the limit $\frac{(m+1)^2}{4}$, and approaches to ∞, each sinuous oval approaches more nearly to the circular arc which separates the two regions d, e, which contains the sinuous oval.

Finally, $\lambda = 0$, the curve is the circle twice repeated.

ON THE PROPERTIES OF THE $\Delta^m 0^n$ CLASS OF NUMBERS AND OF OTHERS ANALOGOUS TO THEM, AS INVESTIGATED BY MEANS OF REPRESENTATIVE NOTATION.

By the Rev. J. BLISSARD.

1. If the differences of the $m+1$ quantities, viz. $(m+r)^n$, $(m+r-1)^n$, ... r^n be successively taken until we arrive at a single expression, the function so obtained is, according to the received notation, symbolically denoted by $\Delta^m r^n$, so that

$$\Delta^m r^n = (m+r)^n - \frac{m}{1}(m+r-1)^n + \frac{m(m-1)}{1.2}(m+r-2)^n - \&c.$$

Hence, if $r = 0, 1, 2$, &c., we have

$$\Delta^m 0^n = m^n - \frac{m}{1}(m-1)^n + \frac{m(m-1)}{1.2}(m-2)^n - \&c.,$$

$$\Delta^m 1^n = (m+1)^n - \frac{m}{1}(m)^n + \frac{m(m-1)}{1.2}(m-1)^n - \&c.,$$

and so on.

This notation is highly significant, and as long as m alone is made to vary and n remains constant, the inconvenience of it as a working notation is not considerable. For in this case Δ is worked with as a representative quantity, and 0^n, 1^n, &c. are mere adjuncts or appendages to be applied throughout every process. But when n is made to vary, Δ^m becomes the adjunct, and the digits or quantities 0, 1, 2, &c. are worked with as representative quantities, and thus cases will perpetually occur in which the same digit will be used in the same function, and often in the same term in two entirely different senses, *i.e.* both naturally, and non-naturally, or symbolically. Thus, for instance, it will be shewn that $fx - f^0 = \Delta^0 f\Delta(1+x)0$, and hence, expanding by Maclaurin's theorem,

$$\frac{d^n fx}{dx^n}(x = {}^0) = f\Delta . 0(0-1)(0-2)...(0-n+1).$$

In these equations the zero digit in f^0, Δ^0 and $\frac{d^n fx}{dx^n}(x = {}^0)$ is to be taken naturally, and in $(1+x)0$ and $f\Delta.0(0-1)...(0-n+1)$

it is to be taken symbolically. A more marked case is the following. Using the $\Delta^m 1^n$ notation, it will be shewn that*

$$\Delta^m 1\,(1-1)(1-2)\ldots(1-n+1)=0,\ (n \gtrless m) \text{ and } = 1.2\ldots m\ (n=m).$$

Here in each of the factors $(1-1)$, $(1-n+1)$, the unit digit is used in two entirely different senses.

Again, it will be shewn that

$\Delta^m\,(p+q)^n$ where p is symbolical and q natural;

$= \Delta^m\,(p+q)^n$ where q is symbolical and p natural;

$= \Delta^m\,(p+q)^n$ where $p+q$, combined as one quantity, is symbolical.

Here the notation utterly fails and loses all power of distinction. All the inconvenience and defectiveness of this notation may be simply remedied as follows:

Instead of $\Delta^m r^n$, assume

$$\Delta^m z_r^{\,n} = (m+r)^n - \frac{m}{1}(m+r-1)^n + \&c.,$$

then when r vanishes or becomes 1.2, &c., we have $\Delta^m z^n$, $\Delta^m z_1^{\,n}$, $\Delta^m z_2^{\,n}$, &c., instead of $\Delta^m 0^n$, $\Delta^m 1^n$, $\Delta^m 2^n$, &c., and the above ambiguous equations become clearly defined as follows:

$$fx - f0 = \Delta^n f\Delta\,(1+x)^z,\ \frac{d^n fx}{dx^n}\,(x=0) = f\Delta\,.\,z\,(z-1)\ldots(z-n+1),$$

$$\Delta^m z_1\,(z_1-1)\,(z_1-2)\ldots(z_1-n+1) = 0,\ \&c.,$$

$$\Delta^m\,(z_p+q)^n = \Delta^m\,(p+z_q)^n = \Delta^m z^n{}_{p+q}.$$

In the final result of any process, the ordinary or received notation might be again substituted.

2. Retaining the received notation when m alone is made to vary, in

$$\Delta^m 0^n = m^n - \frac{m}{1}(m-1)^n + \frac{m\,(m-1)}{1.2}(m-2)^n - \&c.,$$

let $m=0$; therefore $\Delta^0 0^n = 0$, also let u be the representative of the quantities 1^n, $2^n \ldots m^n$, so that $u^m = m^n$; therefore $u^0 = 0$. Hence

$$\Delta^m 0^n = u^m - \frac{m}{1}u^{m-1} + \frac{m\,(m-1)}{1.2}u^{m-2} - \&c. = u^0\,(u-1)^m.$$

* In these cases, as long as we work with this notation, our only remedy seems to be to make the digits when employed symbolically of larger size than when used naturally.

Hence, as will be seen, on expanding by Taylor's Theorem, $\Delta^0 f(h+\Delta x) = u^0 f\{h+(u-1)\,x\}$. And since this holds whatever h may be, &c., therefore when $h=0$, we have (more concisely)

$$\Delta^0 f\Delta \,.\, 0^n = u^0 f(u-1) \text{ .... (I).}$$

By varying the form of $f\Delta$, we shall obtain a large supply of interesting results, some of which are likely to be new.

[1] Let $f\Delta = (1+\Delta)^m$; therefore $f(u-1) = u^m$, and

$$\Delta^0 (1+\Delta)^m . 0^n = u^m = m^n \text{ .... } (a).$$

This equation is usually given under the form

$$\{(1+\Delta)^m - 1\}\, 0^n = m^n.$$

It will be seen that the effect of the factor Δ^0 on $(1+\Delta).^m 0^n$ is to do away with the first term in the expression of $(1+\Delta)^m$.

[2] Let $f\Delta = \Delta^m (1+\Delta)^r$; therefore

$$f(u-1) = u^r (u-1)^m = u^{m+r} - \frac{m}{1} u^{m+r-1}, \text{ \&c.}$$

$$= (m+r)^n - \frac{m}{1}(m+r-1)^{n-1} + \text{\&c.} = \Delta^m r^n;$$

therefore $$\Delta^m (1+\Delta)^r\, 0^n = \Delta^m r^n \text{ .... } (b).$$

From this formula it is evident that any function of the $\Delta^m r^n$ class of numbers can always be expressed in terms of the $\Delta^m 0^n$ numbers.

[3] Let $f\Delta = \left(\dfrac{\Delta}{1+\Delta}\right)^m$; therefore $f(u-1) = \left(\dfrac{u-1}{u}\right)^m$, and

$$\Delta^0 \left(\frac{\Delta}{1+\Delta}\right)^m 0^n = u^0 \left(\frac{u-1}{u}\right)^m = u^0 \left\{1 - \frac{m}{1} u^{-1} + \frac{m(m+1)}{1.2} u^{-2} - \text{\&c.}\right\}$$

$$= -\frac{m}{1}(-1)^n + \frac{m(m-1)}{1.2}(-2)^n - \text{\&c.},$$

which reversing the order of the series, becomes

$$(-1)^{m+n} \left\{m^n - \frac{m}{1}(m-1)^n, \text{ \&c.}\right\} = (-1)^{m+n} \Delta^m 0^n,$$

i.e. $$\left(\frac{\Delta}{1+\Delta}\right)^m 0^n = (-1)^{m+n} \Delta^m 0^n,$$

or $$\left\{\Delta^m - \frac{m}{1}\Delta^{m+1} + \frac{m(m+1)}{1.2}\Delta^{m+2} - \text{\&c.}\right\} 0^n = (-1)^{m+n} \Delta^m 0^n$$

.... (c).

[4] Let $f\Delta = \Delta^m (2+\Delta)^m$, then

$$\Delta^m (2+\Delta)^m 0^n = u^0 (u^2 - 1)^m$$

$$= u^{2m} - \frac{m}{1} u^{2m-2} + \frac{m(m-1)}{1.2} u^{2m-4} - \&c.$$

$$= (2m)^n - \frac{m}{1}(2m-2)^n + \frac{m(m-1)}{1.2}(2m-4)^n - \&c.,$$

i.e. $2^m \Delta^m \left(1+\frac{\Delta}{2}\right)^m 0^n$

$$= 2^n \left\{ m^n - \frac{m}{1}(m-1)^n + \frac{m(m-1)}{1.2}(m-2)^n - \&c. \right\} = 2^n \Delta^m 0^n;$$

therefore $$\Delta^m \left(1+\frac{\Delta}{2}\right)^m 0^n = 2^{n-m} \Delta^m 0^n \text{ .... } (d).$$

[5] If $\Sigma^m 0^n$ be used to denote the *sums* of the $m+1$ quantities m^n, $(m-1)^n$, ... 1^n, 0^n, taken successively until there results a single expression, then

$$\Sigma^m 0^n = m^n + \frac{m}{1}(m-1)^n + \frac{m(m-1)}{1.2}(m-2)^n, \&c.$$

$$= u^m + \frac{m}{1} u^{m-1} + \frac{m(m-1)}{1.2} u^{m-2}, \&c. = u^0 (u+1)^m.$$

Also in (I), let $f\Delta = (2+\Delta)^m$, then

$$\Delta^0 (2+\Delta)^m = u^0 (u+1)^m,$$

i.e. $$\Sigma^m 0^n = \Delta^0 (2+\Delta)^m 0^n \text{ .... } (e).$$

Hence, any function of the $\Sigma^m 0^n$ numbers can always be expressed in terms of the $\Delta^m 0^n$ numbers.

Again, $\Delta^0 (1+\Delta)^r (2+\Delta)^m 0^n = u^r (u+1)^m$

$$= u^{m+r} + \frac{m}{1} u^{m+r-1} + \frac{m(m-1)}{1.2} u^{m+r-2}, \&c.$$

$$= (m+r)^n + \frac{m}{1}(m+r-1)^n + \frac{m(m-1)}{1.2}(m+r-2)^n, \&c.,$$

which if symbolically expressed analogously to the $\Delta^m r^n$ notation would be denoted by $\Sigma^m r^n$. Hence, any function of this class of numbers can always be expressed in terms of the $\Delta^m 0^n$ numbers.

3. In (I), let $f\Delta$ become $f(x+h+h\Delta)$, then

$$\Delta^0 f(x+h+h\Delta) = u^0 f(x+hu),$$

and expanding by Taylor's Theorem and putting $f_r x$ for $\frac{d^r fx}{dx^r}$, we have

$$\left\{f_1(x+h)\,h\Delta + f_2(x+h)\frac{h^2\Delta^2}{1.2} + f_3(x+h).\frac{h^3\Delta^3}{1.2.3}, \text{ \&c.}\right\} 0^n$$

$$= f_1 x.hu + f_2 x.\frac{h^2u^2}{1.2} + f_3 x.\frac{h^3u^3}{1.2.3}, \text{ \&c.}$$

$$= f_1 x.1^{n-1}h + f_2 x.2^{n-1}h^2 + f_3 x.3^{n-1}.\frac{h^3}{1.2} + f_4 x.4^{n-1}\frac{h^4}{1.2.3}, \text{ \&c.}$$

Now for $f_1 x$, $f_2 x$, $f_3 x$, &c. put fx, $f_1 x$, $f_2 x$, &c., then (dividing by h)

$$\left\{f(x+h)\,\Delta + f_1(x+h)\frac{h\Delta^2}{1.2} + f_2(x+h).\frac{h^2\Delta^3}{1.2.3}, \text{ \&c.}\right\} 0^n$$

$$= 1^{n-1}fx + 2^{n-1}f_1 x.\frac{h}{1} + 3^{n-1}fx^n.\frac{h^2}{1.2} + \text{\&c.} \;\ldots\ldots\; \text{(II)}.$$

COR. When $n=1$, we have

$$f(x+h) = fx + f_1 x.\frac{h}{1} + f_2 x.\frac{h^2}{1.2}, \text{ \&c.},$$

which is Taylor's Theorem. Hence formula (II) may be regarded as a kind of generalization of that theorem. The application of this formula will yield some interesting results.

[1] Let $fx = \log x$, then

$$\left\{\log(x+h)\,\Delta + \frac{h}{x+h}\cdot\frac{\Delta^2}{1.2} - \left(\frac{h}{x+h}\right)^2\frac{\Delta^3}{2.3} + \ldots\right\} 0^n$$

$$= \log x + \frac{2^{n-1}}{1}\cdot\frac{h}{x} - \frac{3^{n-1}}{2}\cdot\left(\frac{h}{x}\right)^2 + \frac{4^{n-1}}{3}\left(\frac{h}{x}\right)^3 - \text{\&c.}$$

Hence, substituting x for $\frac{h}{x}$,

$$\log(x+1) + \left\{\frac{x}{1+x}\cdot\frac{\Delta^2}{1.2} - \left(\frac{x}{1+x}\right)^2\cdot\frac{\Delta^3}{2.3} + \text{\&c.}\right\} 0^n$$

$$= 2^{n-1}x - \frac{3^{n-1}}{2}x^2 + \frac{4^{n-1}}{3}x^3 - \text{\&c.} \;\ldots\ldots\ldots\; (f).$$

In this equation we may substitute for x any representative quantity. For x put Δ, applying to it throughout the process the adjunct 0^m; then

$$\log(1+\Delta)\,0^m + \left(\frac{\Delta}{1+\Delta}\right)0^m\frac{\Delta^2 0^n}{1.2} - \left(\frac{\Delta}{1+\Delta}\right)^2 0^m.\frac{\Delta^3 0^n}{2.3} + \ldots$$

$$= \left(2^{n-1}\Delta - \frac{3^{n-1}}{2}.\Delta^2 + \frac{4^{n-1}}{3}.\Delta^3 - \&c.\right)0^m.$$

But (as will be shewn) $\log(1+\Delta)\,0^m = 0\,(m>1)$, and (see Art. 2),

$$\left(\frac{\Delta}{1+\Delta}\right)^r 0^m = (-1)^{m+r}\Delta^r 0^m;$$

$$\text{therefore } (-1^{m+1})\left\{\frac{\Delta 0^m}{1}.\frac{\Delta^2 0^n}{2} + \frac{\Delta^2 0^m}{2}.\frac{\Delta^3 0^n}{3} + \frac{\Delta^3 0^m}{3}.\frac{\Delta^4 0^n}{4}, \&c.\right\}$$

$$= \left(2^{n-1}\Delta - \frac{3^{n-1}}{2}\Delta^2 + \frac{4^{n-1}}{3}\Delta^3 - \&c.\right)0^m.$$

Ex. $(m=3,\ n=4)$, then

$$\frac{\Delta^2 0^4}{2} + \frac{\Delta^2 0^3}{2}.\frac{\Delta^3 0^4}{3} + \frac{\Delta^3 0^3}{3}.\frac{\Delta^4 0^4}{4} = 2^3 - \frac{3^3}{2}\Delta^2 0^3 + \frac{4^3}{2}\Delta^3 0^3,$$

i.e. $$7 + 36 + 12 = 8 - 81 + 128.$$

[2] In (II) let $h=1$ and $fx = x\log x$, then, after reduction, we obtain

$$\frac{3^{n-1}}{1.2}.\frac{1}{x} - \frac{4^{n-1}}{2.3}.\frac{1}{x^2} + \frac{5^{n-1}}{3.4}.\frac{1}{x^3} - \&c.$$

$$= (2^{n-1}+x)\log\left(1+\frac{1}{x}\right) - 1 + \frac{1}{1+x}.\frac{\Delta^3 0^n}{1.2.3}$$

$$- \frac{1}{(1+x)^2}.\frac{\Delta^4 0^n}{2.3.4} + \frac{1}{(1+x)^3}.\frac{\Delta^5 0^n}{3.4.5} - \&c.$$

For x put $\frac{1}{x}$, then

$$\frac{3^{n-1}}{1.2}x - \frac{4^{n-1}}{2.3}x^2 + \&c. = 2^{n-1}\log(1+x) + \frac{1}{x}\log(1+x) - 1$$

$$+ \frac{x}{1+x}\frac{\Delta^3 0^n}{1.2.3} - \left(\frac{x}{1+x}\right)^2\frac{\Delta^4 0^n}{2.3.4} + \&c. \ldots\ldots\ldots\ldots(g).$$

Now for x put Δ, applying the adjunct 0^m, then

$$\left(\frac{3^{n-1}}{1.2}\Delta - \frac{4^{n-1}}{2.3}\Delta^2 + \frac{5^{n-1}}{3.4}\Delta^3 - \&c.\right)0^m$$

$$= 2^{n-1}\log(1+\Delta)\,0^m + \frac{1}{\Delta}\log(1+\Delta)\,0^m - \Delta^0 0^m$$

$$+\left(\frac{\Delta}{1+\Delta}\right)0^m.\frac{\Delta^3 0^n}{1.2.3} - \left(\frac{\Delta}{1+\Delta}\right)^2 0^m.\frac{\Delta^4 0^n}{2.3.4} + \ldots.$$

But $\log(1+\Delta)\,0^m = 0$, $\frac{1}{\Delta}\log(1+\Delta)\,0^m = B_m$ (as will be shewn) where B is the representative of Bernoulli's numbers, $\Delta^0 0^n = 0$ and $\left(\frac{\Delta}{1+\Delta}\right)^r 0^m = (-1)^{m+r}\Delta^r 0^m$. Hence

$$\left(\frac{3^{n-1}}{1.2}\Delta - \frac{4^{n-1}}{2.3}\Delta^2 + \frac{5^{n-1}}{3.4}\Delta^3 - \&c.\right)0^m$$

$$= B_m + (-1)^{m-1}\left\{\Delta 0^m.\frac{\Delta^3 0^n}{1.2.3} + \Delta^2 0^m.\frac{\Delta^4 0^n}{2.3.4} + \frac{\Delta^3 0^m.\Delta^5 0^n}{3.4.5} + \&c.\right\}.$$

Ex. $(m=3,\ n=5)$,

$$\frac{81}{2} - 256 + \frac{625}{2} = B_3\,(=0) + 25 + 60 + 12.$$

In the same way, if in (II), when $h=1$, *i.e.* in

$$\left\{f(x+1)\,\Delta + f(x+1)\,\frac{\Delta^2}{1.2} + \&c.\right\}0^n$$

$$= 1^{n-1}fx + 2^{n-1}f_1x + \frac{3^{n-1}}{1.2}f_2x + \&c.$$

we put $$fx = x^2\log x,\ \ x^3\log x,\ \&c.,$$

a succession of results will be produced of a similar character with the above, and which probably cannot be arrived at by any other method.

In such equations as above involving x, we may vary our results almost indefinitely by substituting different representative quantities for x. Thus in (f), viz.

$$\log(1+x) + \left\{\frac{x}{1+x}.\frac{\Delta^2}{1.2} + \left(\frac{x^2}{1+x}\right)\frac{\Delta^3}{2.3},\ \&c.\right\}0^n$$

$$= 2^{n-1}x - 3^{n-1}\frac{x^2}{2} + 4^{n-1}\frac{x^3}{3} - \&c.,$$

by putting P for x, where P is the representative of the natural numbers, that is, $P^m = m$, then since

$$\log(1+P) = P - \frac{P^2}{2} \times \frac{P^3}{3} - \&c. = 1 - 1 + 1 - \&c. = \frac{1}{2},$$

$$\left(\frac{P}{1+P}\right)^r = P^r - rP^{r+1} + \frac{r(r+1)}{1.2} P^{r+2} - \&c.$$

$$= r - r(r+1) + \frac{r(r+1)(r+2)}{1.2} - \&c. = \frac{r}{2^{r+1}};$$

we have

$$\frac{1}{2} + \left(\frac{1}{2^2} \cdot \frac{\Delta^2}{1.2} - \frac{2}{2^3} \cdot \frac{\Delta^3}{2.3} + \frac{3}{2^4} \cdot \frac{\Delta^4}{3.4} - \&c.\right) 0^n$$

$$= 2^{n-1} - 3^{n-1} + 4^{n-1} - \&c.$$

Hence since

$$2^{n-1} - 3^{n-1} + 4^{n-1}, \&c. = 1 - \frac{2^n - 1}{n} B_n$$

(as can be shewn); therefore

$$\left(\frac{1}{2} \cdot \frac{\Delta^2}{2^2} - \frac{1}{3} \cdot \frac{\Delta^3}{2^3} + \frac{1}{4} \cdot \frac{\Delta^4}{2^4} - \&c.\right) 0^n = \frac{1}{2} - \frac{2^n - 1}{n} B_n.$$

Ex. $(n=4)$ $\frac{1}{2} \cdot \frac{14}{4} - \frac{1}{3} \cdot \frac{36}{8} + \frac{1}{4} \cdot \frac{24}{16} = \frac{1}{2} + \frac{15}{4} \cdot \frac{1}{30} = \frac{5}{8}.$

In (g) put $\frac{1}{B+1}$ for x, then

$$\frac{3^{n-1}}{1.2} \cdot \frac{1}{B+1} - \frac{4^{n-1}}{2.3} \cdot \frac{1}{(B+1)^2} + \frac{5^{n-1}}{3.4} \cdot \frac{1}{(B+1)^3} - \&c.$$

$$= (2^{n-1} + B + 1)\{\log(B+2) - \log(B+1)\}$$

$$- 1 + \frac{1}{B+2} \frac{\Delta^3 0^n}{1.2.3} - \frac{1}{(B+2)^2} \frac{\Delta^4 0^n}{2.3.4} + \ldots.$$

But $\frac{1}{(B+1)^m} = mS_{m+1}$, $\frac{1}{(B+2)^m} = m(S_{m+1} - 1) = mD_{m+1}$,

$$2^{n-1}\{\log(B+2) - \log(B+1)\} = 2^{n-1},$$

$$(B+1)\{\log(B+2) - \log(B+1)\} = \gamma.$$

Hence

$$\frac{3^{n-1}}{2} S_2 - \frac{4^{n-1}}{3} S_3, \&c. = 2^{n-1} + \gamma - 1 + D_2 \cdot \frac{\Delta^3 0^n}{1.2.3} - D_3 \cdot \frac{\Delta^4 0^n}{2.3.4}, \&c.$$

For S_2, S_3, &c. put D_2+1, D_3+1, &c., then, since from (f),

$$\frac{3^{n-1}}{2}-\frac{4^{n-1}}{3}+\frac{5^{n-1}}{4}-\text{\&c.}$$
$$=2^{n-1}-\log 2-\left(\frac{1}{1.2}\cdot\frac{\Delta^2}{2}-\frac{1}{2.3}\cdot\frac{\Delta^3}{2^2}+\frac{1}{3.4}\cdot\frac{\Delta^4}{2.3}-\text{\&c.}\right)0^n,$$

we have $\dfrac{3^{n-1}}{2}D_2-\dfrac{4^{n-1}}{3}D_3+\dfrac{5^{n-1}}{4}D_4-\text{\&c.}$

$$=\gamma+\log 2-1+\left(\frac{1}{1.2}\cdot\frac{\Delta^2}{2}-\frac{1}{2.3}\cdot\frac{\Delta^3}{2^2}+\frac{1}{3.4}\cdot\frac{\Delta^4}{2^3}-\text{\&c.}\right)0^n$$
$$+\frac{\Delta^3 0^n}{2.3}D_2-\frac{\Delta^4 0^n}{3.4}D_3+\frac{\Delta^5 0^n}{4.5}D_4-\text{\&c.}\ \ldots\ldots\ldots\ldots\ (h).$$

[3] In (II) put $fx=\varepsilon^x$, then, dividing by ε^x, we have

$$\varepsilon^h\left(\Delta+h\frac{\Delta^2}{1.2}+h^2\frac{\Delta^3}{1.2.3}+\ldots\right)0^n=1^{n-1}+2^{n-1}\frac{h}{1}+3^{n-1}\frac{h^2}{1.2},\ \text{\&c.}$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(i).$$

4. In equation (a), viz. $\Delta^0(1+\Delta)^m 0^n=m^n$, we may put for m any representative quantity as P, and hence

$$\Delta^0(1+\Delta)^P 0^n=P^n.$$

The factor Δ^0 may always be withdrawn when no term, which does not involve Δ, enters in the expansion of $(1+\Delta)^P$. Our object is to assume for $(1+x)^P$ such a function of x, that the value of P^n may be capable of being determined. For this purpose it is necessary that the function so assumed should be capable of development in terms of ascending powers of x.

[1] Let $(1+x)^P=\log^m(1+x)$, then putting ε^x for $1+x$, $\varepsilon^{Px}=x^m$. Hence expanding and equating powers of x, $P^n=0$ unless $n=m$, in which case $P^m=1.2\ldots m$; therefore

$$\log^m(1+\Delta)\,0^n=P^n=0\ (n>m)\ \ldots\ldots\ldots\ldots(1),$$

and $$\log^m(1+\Delta)\,0^m=1.2\ldots m;$$

therefore $$\Delta^m 0^m=1.2\ldots m.$$

Ex. $(m=2)$, then

$\log^2(1+\Delta)\,0^n$

$$=2\left\{\frac{\Delta^2}{2}-\left(1+\frac{1}{2}\right)\frac{\Delta^3}{3}+\left(1+\frac{1}{2}+\frac{1}{3}\right)\frac{\Delta^4}{4}-\text{\&c.}\right\}0^n=0\ (n\gtrless 2),$$

let $n=4$, then

$$\Delta^2 0^4 - \Delta^3 0^4 + \frac{11}{12}\,\Delta^4 0^4 = 14 - 36 + 22 = 0.$$

[2] Let $\qquad (1+x)^P = (1+x)^r \log^m (1+x)$,

then $\quad \varepsilon^{Px} = \varepsilon^{rx}.x^m \quad$ and $\quad P^n = n\,(n-1)\ldots(n-m+1)\,r^{n-m}$.

Hence $(1+\Delta)^r \log^m (1+\Delta)\,0^n = P^n$

$$= n\,(n-1)\ldots(n-m+1)\,r^{n-m}\ldots\ldots\ldots\ldots(2).$$

Ex. $(m=2,\ r=3,\ n=4)$, then

$$(1+\Delta)^3 \left(\Delta^2 - \frac{\Delta^3}{3} + \frac{\Delta^4}{4}\right) 0^4 = 4.3.3^2 = 108,$$

i.e. $\qquad \Delta^2 0^4 + 2\Delta^3 0^4 + \frac{11}{12}\,\Delta^4 0^4\ (= 14 + 72 + 22) = 108.$

[3] Let $\qquad (1+x)^P = x^r\,(1+x)^p \log^{m} (1+x)$,

then $\varepsilon^{Px} = (\varepsilon^x - 1)^r.\varepsilon^{px}.x^m$

$$= x^m \left\{\varepsilon^{(p+r)x} - \frac{r}{1}\,\varepsilon^{(p+r-1)x} + \frac{r\,(r-1)}{1.2}\,\varepsilon^{(p+r-2)x} - \&c.\right\}.$$

Hence expanding and equating coefficients

$$\frac{P^n}{1.2\ldots n} = \frac{1}{1.2\ldots(n-m)}$$

$$\times \left\{(p+r)^n - \frac{r}{1}\,(p+r-1)^n + \frac{r\,(r-1)}{1.2}\,(p+r-2)^n - \&c.\right\}$$

$$= \frac{1}{1.2\ldots(n-m)}\,\Delta^r\,(1+\Delta)^p\,0^{n-m},$$

(see (*b*), Art. 2). Therefore

$\Delta^r\,(1+\Delta)^p \log^m (1+\Delta)\,0^n$

$$= n\,(n-1)\ldots(n-m+1)\,\Delta^r\,(1+\Delta)^p\,0^{n-m}\ldots\ldots\ldots(3).$$

If $p=0$,

$$\Delta^r \log^m (1+\Delta)\,0^n = n\,(n-1)\ldots(n-m+1)\,\Delta^r 0^{n-m}\ldots(4).$$

In (4) for r put $-m$ and for n put $n+m$, then

$$\Delta^{-m} 0^n = \frac{\Delta^0}{(n+1)\,(n+2)\ldots(n+m)}\cdot\left(1 - \frac{\Delta}{2} + \frac{\Delta^2}{3} - \&c.\right)^m 0^{m+n}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5).$$

But (see *Quarterly Journal*, No. 18, p. 192)

$$\Delta^{-m}0^n = \frac{(-1)^{m-1}}{\Gamma m}\left(\frac{B_{n+m}}{n+m} + q_1\frac{B_{n+m-1}}{n+m-1} + \ldots + q_{m-1}\frac{B_{n+1}}{n+1}\right),$$

where B is the representative of Bernoulli's numbers and

$$x(x+1)(x+2)\ldots(x+m-1) = x^m + q_1x^{m-1} + q_2x^{m-2} + \ldots + q_{m-1}x.$$

Hence $\dfrac{1}{(n+1)\ldots(n+m)}\,\Delta^0\left(1-\dfrac{\Delta}{2}+\dfrac{\Delta^2}{3}-\&c.\right)^m 0^{m+n}$

$$= \frac{(-1)^{m-1}}{\Gamma m}\left(\frac{B_{n+m}}{n+m} + q_1\frac{B_{n+m-1}}{n+m-1} + \ldots + q_{m-1}\frac{B_{n+1}}{n+1}\right).$$

Ex. 1. Let $m = 1$, then

$$\frac{1}{n+1}\left(1-\frac{\Delta}{2}+\frac{\Delta^2}{3}-\&c.\right)0^{n+1} = \frac{1}{n+1}B_{n+1},$$

a result which can be obtained in another way as follows:
In $\Delta^0(1+\Delta)^m\,0^n = m^n$ put B for m, then

$$\Delta^0(1+\Delta)^B\,0^n = B^n = B_n.$$

But $$(1+x)^B = \frac{1}{x}\log(1+x);$$

therefore $$\frac{1}{\Delta}\log(1+\Delta)\,0^n = B_n \ldots\ldots\ldots\ldots\ldots\ldots(h).$$

Ex. 2. Let $m = 2$, then

$$\frac{B_{n+2}}{n+2} + \frac{B_{n+1}}{n+1} = \frac{1}{(n+1)(n+2)}\,\Delta^0\left(1-\frac{\Delta}{2}+\frac{\Delta^2}{3}-\&c.\right)^2 0^{n+2};$$

therefore $(n+1)B_{n+2} + (n+2)B_{n+1}$

$$= -2\left\{\left(1+\frac{1}{2}\right)\frac{\Delta}{3} - \left(1+\frac{1}{2}+\frac{1}{3}\right)\frac{\Delta^2}{4} + \left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right)\frac{\Delta^3}{5} - \&c.\right\}0^{n+2},$$

let $n = 4$, then

$$3B_4 + 4B_5 = \left(-\frac{1}{10}\right) = -1 + \frac{11}{12}\Delta^2 0^4 - \frac{5}{6}\Delta^3 0^4 + \frac{137}{180}\Delta^4 0^4$$

$$= -1 + \frac{77}{6} - 30 + \frac{274}{15}.$$

5. Let ${}_rC_n$ denote the sum of the products of the n quantities, viz. 1, 2, ... n taken r together, then

$$\left(1-\frac{x}{2}+\frac{x^2}{3}-\&c.\right)^m = 1 - {}_1C_m \cdot \frac{x}{m+1}$$

$$+ {}_2C_{m+1} \cdot \frac{x^2}{(m+1)(m+2)} - {}_3C_{m+2} \cdot \frac{x^3}{(m+1)(m+2)(m+3)} + \ldots .$$

Hence, from (III),

$$\Delta^{-m}0^n = -\frac{1}{(n+1)\ldots(n+m)}$$

$$\times \left\{ {}_1C_m \frac{\Delta}{m+1} - {}_2C_{m+1} \cdot \frac{\Delta^2}{(m+1)(m+2)}, \&c. \right\} 0^{m+n},$$

which therefore

$$= \frac{(-1)^{m-1}}{1.2\ldots(m-1)} \left(\frac{B_{n+m}}{n+m} + {}_1C_{m-1} \cdot \frac{B_{n+m-1}}{n+m-1} \right.$$

$$\left. + {}_2C_{m-1} \cdot \frac{B_{n+m-2}}{n+m-2} + \ldots + {}_{m-1}C_{m-1} \cdot \frac{B_{n+1}}{n+1} \right).$$

But, if in

$$\Delta^m 0^n = m^n - \frac{m}{1}(m-1)^{n-1} + \frac{m(m-1)}{1.2}(m-2)^{n-2} - \&c.,$$

$-m$ be put for m, then

$$\Delta^{-m}0^n z (-1)^n \left\{ m^n + \frac{m}{1}(m+1)^n + \frac{m(m+1)}{1.2}(m+2)^n, \&c. \right\},$$

which is evidently infinite in value, whereas the values of $\Delta^{-m}0^n$ above given are of finite value. This anomaly is analogous to one already pointed out (see *Quarterly Journal*, No. 22, p. 176). Thus, if $D_m = \frac{1}{2^m} + \frac{1}{3^m} + \&c.$, *ad infinitum*, then D_0, D_{-1}, D_{-2}, &c. all become infinite in value. But these quantities are found to have finite values, viz. $D_0 = -\frac{3}{2}$, D_{-1}, D_{-2}, &c. severally $= -1$, which values satisfy all the equations in which they enter. These anomalies are (I cannot but think) capable of an explanation, the principle of which will apply to all analogous cases.

6. In (2), viz.

$$(1+\Delta)^r \log^m (1+\Delta)\, 0^n = n(n-1)\ldots(n-m+1)\, r^{n-m},$$

let $m=1$, then

$$(1+\Delta)^r \log(1+\Delta)\, 0^n = nr^{n-1}.$$

ELEMENTARY MATHEMATICAL WORKS

BY THE

RIGHT REV. J. W. COLENSO, D.D.

LORD BISHOP OF NATAL.

ARITHMETIC designed for the use of SCHOOLS; to which is added a Chapter on DECIMAL COINAGE. New Edition, thoroughly revised, with the addition of Notes and Examination Papers. 12mo. 4*s.* 6*d.*

KEY to COLENSO'S ARITHMETIC for SCHOOLS; containing full SOLUTIONS of all the Exercises: adapted to the above Revised and Enlarged Edition of 1864. By the Rev. J. HUNTER, M.A. 12mo. 5*s.*

COLENSO'S ELEMENTARY ARITHMETIC, for the use of NATIONAL, ADULT, and COMMERCIAL SCHOOLS; consisting of a Text-Book of Arithmetic, accompanied by a Series of easy and amusing Examples arranged progressively in order of difficulty. 18mo. 1*s.* 9*d.*; or with ANSWERS, 2*s.* 3*d.*

COLENSO'S ELEMENTARY ARITHMETIC for National Schools is divided into FIVE PARTS, which are sold separately as follows:—

I. TEXT BOOK, price 6*d.*
II. EXAMPLES, PART I. *Simple Arithmetic*, price 4*d.*
III. EXAMPLES, PART II. *Compound Arithmetic*, price 4*d.*
IV. EXAMPLES, PART III. *Fractions, Decimals, Duodecimals*, &c. price 4*d.*
V. ANSWERS to the EXAMPLES, with SOLUTIONS of the more difficult QUESTIONS, price 1*s.*

COLENSO'S ARITHMETICAL TABLES, on a Card, Price 1*d.*

THE ELEMENTS of ALGEBRA, designed for the use of SCHOOLS, PART I. containing the Simpler Parts of the Science, suitable for general School purposes, and as required for the attainment of an ordinary B.A. degree. 12mo. 4*s.* 6*d.* KEY, 5*s.*

HUNTER'S EXAMINATION-QUESTIONS ON COLENSO'S ALGEBRA, PART I. with Graduated Exercises, Problems, &c. 12mo. 2*s.* 6*d.*

COLENSO'S ELEMENTS OF ALGEBRA, designed for the use of SCHOOLS, PART II. adapted to suit the wants of more advanced Students; containing the higher parts of the subject, with an APPENDIX of more difficult Miscellaneous Examples and Equation Papers. 12mo. 6*s.* KEY, 5*s.*

COLENSO'S MISCELLANEOUS EXAMPLES AND EQUATION PAPERS, from PARTS I. and II. of the *Algebra*; with the ANSWERS. 12mo. 2*s.* 6*d.*

COLENSO'S ELEMENTS OF ALGEBRA, adapted for the use of NATIONAL and ADULT SCHOOLS; containing numerous easy Examples and Questions under every Rule, with a Selection of Miscellaneous Exercises for Practice. 18mo. 1*s.* 6*d.* KEY, 2*s.* 6*d.*

COLENSO'S ELEMENTS OF ALGEBRA, adapted for the use of TEACHERS and STUDENTS in the UNIVERSITY; being a Large-Paper Edition of the above. 8vo. 12*s.* 6*d.* KEY, 12mo. 7*s.* 6*d.*

PLANE TRIGONOMETRY, PART I. comprising the Measurement of Lines and Angles, the Numerical Values of the Trigonometrical Ratios; with the use of Logarithms and the Exponential Theorem. 12mo. 3*s.* 6*d.* KEY, 3*s.* 6*d.*

COLENSO'S PLANE TRIGONOMETRY, PART II. comprising the Summation of Series, the Trigonometrical Solution of Equations, and Miscellaneous Problems. 12mo. 2*s.* 6*d.* KEY, 5*s.*

*** The KEYS to the TWO PARTS of the TRIGONOMETRY, together, price 8*s.* 6*d.*

THE ELEMENTS of EUCLID (as studied in the Universities); with a large Collection of GEOMETRICAL PROBLEMS. 18mo. 4*s.* 6*d.*; or with a KEY to the PROBLEMS, 6*s.* 6*d.*

COLENSO'S GEOMETRICAL PROBLEMS and KEY. 18mo. 3*s.* 6*d.*

COLENSO'S GEOMETRICAL PROBLEMS separately, for the use of Schools where other Editions of *Euclid* may be employed. 18mo. 1*s.*

London: **LONGMANS, GREEN, and Co., Paternoster Row.**

No. 30. October, 1866.

THE

QUARTERLY JOURNAL

OF

PURE AND APPLIED

MATHEMATICS.

EDITED BY

J. J. SYLVESTER, M.A., F.R.S.,

PROFESSOR OF MATHEMATICS IN THE ROYAL MILITARY ACADEMY, WOOLWICH; AND

N. M. FERRERS, M.A.,

FELLOW AND TUTOR OF GONVILLE AND CAIUS COLLEGE, CAMBRIDGE:

ASSISTED BY

G. G. STOKES, M.A., F.R.S.,

LUCASIAN PROFESSOR OF MATHEMATICS IN THE UNIVERSITY OF CAMBRIDGE

A. CAYLEY, M.A., F.R.S.,

SADLERIAN PROFESSOR OF MATHEMATICS IN THE UNIVERSITY OF CAMBRIDGE; AND

M. HERMITE,

CORRESPONDING EDITOR IN PARIS.

ὅ τι οὐσία πρὸς γένεσιν, ἐπιστήμη πρὸς πίστιν καὶ διάνοια πρὸς εἰκασίαν ἐστ

LONDON:

LONGMANS, GREEN, AND CO.,

PATERNOSTER ROW.

1866.

W. METCALFE, PRINTER, | *PRICE FIVE SHILLINGS.* | GREEN STREET, CAMBRIDGE.

CONTENTS.

PAGE

NOTICES TO CORRESPONDENTS.

The following Papers have been received:

Mr. Walton, "On Certain Transformations in the Calculus of Operations;" "On the Symbol of Operation $x\frac{d}{dx}$;" "On the Debility of Large Trees and Animals;" "On Biangular Coordinates;" and "Note on the Lunar Theory."

Professor Cayley, "Specimen Table $M \equiv a^\gamma b^\beta$ (mod. N) for any Prime or Composite Modulus;" "Tables of the Binary Cubic Forms for the Negative Determinants, $\equiv 0$ (Mod. 4), from -4 to -400; and $\equiv 1$ (Mod. 4), from -3 to -99; and for five Irregular Negative Determinants;" "On the Conics which pass through two given Points, and touch two given Lines;" and "On a certain Envelope depending on a Triangle Inscribed in a Circle."

Mr. J. Power, "On the Problem of the Fifteen School Girls."

Mr. Ellis, "Investigation of an Algebraical Formula."

Professor Adolph Steen, "On Linear Differential Equations with Particular Integrals all of the same form."

Professor L. Schläflj, "Solution of a Partial Differential Equation."

Mr. Ferrers, "On an Envelope depending on a Triangle inscribed in a Circle."

Notice to Subscribers.—The first Seven Volumes of the 'Journal,' bound in calf, gilt, or any of the back Numbers, can be had of the Publishers, or at the Printing Office, Green Street, Cambridge.

No. 31 will be published in January, 1867.

Hence, giving to r the values 1, 2, 3, &c., we obtain, after slight reduction,

$$(1)\quad \left\{\frac{\Delta^2}{1.2}-\frac{\Delta^3}{2.3},\ \&c. \pm \frac{\Delta^n}{(n-1)\,n}\right\} 0^n = n-1.$$

$$(2)\quad \left\{\frac{\Delta^3}{1.2.3}-\frac{\Delta^4}{2.3.4},\ \&c. \pm \frac{\Delta^n}{(n-2)(n-1)n}\right\} 0^n = 1+(n-3)\,2^{n-2}.$$

$$(3)\quad \left\{\frac{\Delta^4}{1.2.3.4}-\frac{\Delta^5}{2.3.4.5},\ \&c. \pm \frac{\Delta^n}{(n-3)\ldots n}\right\} 0^n$$
$$= \tfrac{1}{4}\left\{(2n-11)\,3^{n-2}+2^{n+1}-1\right\},$$

and so on.

From the same formula, by making $r=1$ and $m=1.2.3$, &c., we obtain

$$(1)\qquad n = (1+\Delta)\log(1+\Delta)\,0^n.$$

$$(2)\qquad n^2 = (1+\Delta)\left\{\log(1+\Delta)+\log^2(1+\Delta)\right\} 0^n.$$

$$(3)\quad n^3 = (1+\Delta)\left\{\log(1+\Delta)+3\log^2(1+\Delta)+2\log^3(1+\Delta)\right\} 0^n,$$

and generally

$$n^{m+1} = (1+\Delta)\left\{\log(1+\Delta)+{}_1C_m\log^2(1+\Delta)\right.$$
$$\left.+\,{}_2C_m\log^3(1+\Delta)+\ldots+{}_mC_m\log^{m+1}(1+\Delta)\right\} 0^n.$$

8. In (4), viz.

$$\Delta^r\log^m(1+\Delta)\,0^n = n(n-1)\ldots(n-m+1)\,\Delta^r 0^{n-m} = \frac{\Gamma(n+1)}{\Gamma(n-m+1)}\,\Delta^r 0^{n-m},$$

put $-m$ for m, then

$$\Delta^r\log^{-m}(1+\Delta)\,0^n,$$

i.e. $$\Delta^{r-m}\left(1-\frac{\Delta}{2}+\frac{\Delta^2}{3}-\&c.\right)^{-m} 0^n$$

$$= \frac{\Gamma(n+1)}{\Gamma(n+m+1)}\,\Delta^r 0^{n+m} = \frac{1}{(n+1)\ldots(n+m)}\,\Delta^r 0^{m+n} \ldots\ldots (6).$$

When r is $< m$, negative powers of Δ will enter and we shall be able to test the correctness of equation (5).

(1) Let $m=1$, $r=0$, then

$$\frac{1}{\log(1+\Delta)}\,0^n,$$

i.e. $$\Delta^{-1}\left(1-\frac{\Delta}{2}+\frac{\Delta^2}{3}-\&c.\right)^{-1} 0^n = \frac{1}{n+1}\,\Delta^0 0^{n+1} = 0.$$

But $\dfrac{x}{\log(1+x)}$ expanded

$$=1-\frac{x}{2}+\frac{x^2}{12}-\frac{x^3}{24}+\frac{19}{720}x^4-\frac{3}{160}x^5+\&c.$$

Hence $\Delta^{-1}0^n-\dfrac{\Delta^0}{2}0^n+\left(\dfrac{\Delta}{12}-\dfrac{\Delta^2}{24}+\dfrac{19}{720}\Delta^3-\dfrac{3}{160}\Delta^4,\&c.\right)0^n=0,$

also $$\Delta^{-1}0^n=-\frac{B_{n+1}}{n+1} \text{ and } \Delta^0 0^n=0;$$

therefore $$\left(\frac{\Delta}{12}-\frac{\Delta^2}{24}+\frac{19}{720}\Delta^3-\&c.\right)0^n=\frac{B_{n+1}}{n+1}\text{........ (7)}.$$

Ex. $(n=3)$, $\dfrac{1}{12}-\dfrac{1}{4}+\dfrac{19}{120}=\dfrac{B_4}{4}=-\dfrac{1}{120}$.

(2) Let $m=1$, $r=-1$, then

$$\Delta^{-2}\left(1-\frac{\Delta}{2}+\frac{\Delta^2}{12}-\frac{\Delta^3}{24},\&c.\right)0^n=\frac{1}{n+1}\Delta^{-1}0^{n+1}=\frac{B_{n+2}}{(n+1)(n+2)},$$

i.e. $\Delta^{-2}0^n-\dfrac{1}{2}\Delta^{-1}0^n+\dfrac{\Delta^0}{12}0^n-\left(\dfrac{\Delta}{24}-\dfrac{19}{720}\Delta^2,\&c.\right)0^n=\dfrac{B_{n+2}}{(n+1)(n+2)}$.

But $\Delta^{-2}0^n=-\left(\dfrac{B_{n+2}}{n+2}+\dfrac{B_{n+1}}{n+1}\right)$ and $\dfrac{1}{2}\Delta^{-1}0^n=\dfrac{1}{2}\dfrac{B_{n+1}}{n+1}$.
Hence

$$\left(\frac{\Delta}{24}-\frac{19}{720}\Delta^2+\frac{3}{160}\Delta^3-\&c.\right)0^n=-\left(\frac{B_{n+2}}{n+1}+\frac{1}{2}\frac{B_{n+1}}{n+1}\right)\text{...(8)}.$$

Ex. $(n=3)$, $\dfrac{1}{24}-\dfrac{19}{120}+\dfrac{9}{80}=-\dfrac{B_5}{4}-\dfrac{B_4}{8}=\dfrac{1}{240}$.

It is thus evident that the finite values assigned in Art. 5 to $\Delta^{-m}0^n$ satisfy all the equations in which they enter.

9. Returning to equation (a), viz. $\Delta^0(1+\Delta)^m 0^n=m^n$.

(1) Let $m=A$, where $A_n=-\dfrac{2^{n+2}-2}{n+1}B_{n+1}$, then since

$$(1+x)^A=\frac{2}{2+x},$$

$$\Delta^0(1+\Delta)^A 0^n\left\{=\frac{\Delta^0}{1+\frac{\Delta}{2}}0^n=\left(\frac{\Delta}{2}-\frac{\Delta^2}{2^2}+\frac{\Delta^3}{2^3}-\&c.\right)0^n\right\}$$

$$=-A_n=\frac{2^{n+2}-2}{n+1}B_{n+1}\text{............(9)}.$$

(2) For m put $2A$, then

$$\Delta^0(1+\Delta)^{2A}0^n = 2^n A^n = 2^n A_n = 2^{n+1}\frac{(2^{n+1}-1)}{n+1}B_{n+1}.$$

But $(1+\Delta)^{2A} = \dfrac{2}{(\Delta+1)^2+1}$; therefore $\dfrac{2\Delta^0}{\Delta^2+2\Delta+2}\,0^n$ which

$$= \left(\Delta - \frac{\Delta^2}{2} + \frac{\Delta^4}{2^2} - \frac{\Delta^5}{2^2} + \frac{\Delta^7}{2^3} - \frac{\Delta^8}{2^3}, \text{ \&c.}\right)0^n = 2^{n+1}\frac{(2^{n+1}-1)}{n+1}B_{n+1}$$

......................(10).

Ex. $(n=5)$, $1 - 15 + 60 - 30 = 2^6\dfrac{(2^6-1)}{6}B_6 = \dfrac{64.63}{6.42} = 16.$

(3) For m put $3A$, then

$$\Delta^0(1+\Delta)^{3A}0^n = 3^n A^n = 3^n A_n.$$

But
$$(1+x)^{3A} = \frac{2}{(1+x)^3+1};$$

therefore $\dfrac{2\Delta^0}{(\Delta+1)^3+1}\,0^n = 3^n A_n = -3^n.\dfrac{2^{n+2}-2}{n+1}B_{n+1}.$

Hence, after reduction,

$$\frac{2\Delta-\Delta^2-\Delta^3}{1+\Delta^3}\,0^n \text{ which} = (2\Delta-\Delta^2-\Delta^3+2\Delta^4-\Delta^5-\Delta^6+2\Delta^7-\text{\&c.})\,0^n$$

$$= (3^{n+1}-1)(2^{n+1}-1)\frac{B_{n+1}}{n+1} \text{ .... (11).}$$

Ex. $n=5$, then $2 - 30 - 150 + 480 - 120 = \dfrac{728.63}{6.42}$.

(4) For m put $2B$, then

$$\Delta^0(1+\Delta)^{2B}0^n = 2^n B^n = 2^n B_n.$$

But

$$(1+x)^{2B} = \frac{2\log(1+x)}{(1+x)^2-1} = \frac{2\log(1+x)}{x(x+2)} = \frac{\log(1+x)}{x} - \frac{\log(1+x)}{x+2}.$$

Hence
$$\frac{\log(1+\Delta)}{\Delta} - \frac{\log(1+\Delta)}{2+\Delta} = 2^n B_n,$$

and
$$\frac{\log(1+\Delta)}{2+\Delta} = -(2^n-1)B_n \text{ .... (12).}$$

(5) For m put $B+m+1$, then

$$\Delta^0(1+\Delta)^{B+m+1}0^n = (B+m+1)^n.$$

But
$$\Delta^0(1+\Delta)^B = \frac{\log(1+\Delta)}{\Delta},$$

and, as can be shewn,

$$1^n+2^n+\ldots+m^n=\frac{(B+m+1)^{n+1}-(B+1)^{n+1}}{n+1}.$$

Hence, putting $n+1$ for n,

$$\Delta^0(1+\Delta)^{m+1}(1+\Delta)^B 0^{n+1}\{=\Delta^{-1}(1+\Delta)^{m+1}\log(1+\Delta)\,0^{n+1}\}$$
$$=(B+m+1)^{n+1}=B_{n+1}+(n+1)(1^n+2^n+\ldots+m^n).$$

Also, from (3),

$$\Delta^{-1}(1+\Delta)^{m+1}\log(1+\Delta)\,0^{n+1}=(n+1)\,\Delta^{-1}(1+\Delta)^{m+1}\,0^n$$
$$=(n+1)\left\{\Delta^{-1}+(m+1)\Delta^0+\frac{(m+1)m}{1.2}\Delta+\frac{(m+1)\,m(m-1)}{1.2.3}\Delta^2,\ \&c.\right\}0^n.$$

But $\Delta^{-1}0^n=\dfrac{B_{n+1}}{n+1}$. Hence

$$1^n+2^n+\ldots+m^n=\left\{\frac{(m+1)\,m}{1.2}\Delta+\frac{(m+1)\,m(m-1)}{1.2.3}\Delta^2+\&c.\right\}0^n$$
$$\ldots\ldots\ldots\ldots\ldots\ldots(13).$$

Cor. 1. Since $1^n+2^n+\ldots+m^n=\dfrac{(B+1+m)^{n+1}-(B+1)^{n+1}}{n+1}$

$$=m\,(B+1)^n+\frac{n}{2}\,m^2\,(B+1)^{n-1}+\frac{n\,(n-1)}{2.3}\,m^3\,(B+1)^{n-2}+\ldots$$
$$+m^n\,(B+1)+\frac{m^{n+1}}{n+1}$$
$$=mB_n+\frac{n}{2}\,m^2B_{n-1}+\frac{n\,(n-1)}{2.3}\,B_{n-2}+\ldots-B_1m^n+\frac{m^{n+1}}{n+1},$$

(from $(B+1)^n=n$ $(n>1)$ and $B+1=\frac{1}{2}=-B_1$), we have

$$mB_n+\frac{n}{2}\,m^2B_{n-1}+\ldots-B_1m^n+\frac{m^{n+1}}{n+1}$$
$$=\left\{\frac{(m+1)\,m}{1.2}\Delta+\frac{(m+1)\,m\,(m-1)}{1.2.3}\Delta^2,\ \&c.\right\}0^n.$$

Now, put B for m, then

$$B_1B_n+\frac{n}{2}\,B_2B_{n-1}+\frac{n\,(n-1)}{2.3}\,B_3B_{n-2}+\ldots-B_1B_n+\frac{B_{n+1}}{n+1}$$
$$=\left\{\frac{(B+1)\,B}{1.2}\Delta+\frac{(B+1)\,B\,(B-1)}{1.2.3}\Delta^2,\ \&c.\right\}0^n$$
$$=\left\{-\frac{\Delta}{2.3}+\frac{\Delta^2}{3.4}-\frac{\Delta^3}{4.5}+\&c.\right\}0^n\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (14).$$

Hence $\left(\frac{\Delta}{2.3} - \frac{\Delta^2}{3.4} + \frac{\Delta^3}{4.5} - \&c.\right) 0^n = 0$ (n even).

Ex. $(n=4)$, then $\frac{1}{6} - \frac{7}{6} + \frac{9}{5} - \frac{4}{5} = 0$.

COR. 2. In the equation

$$\frac{(B+1+m)^{n+1} - (B+1)^{n+1}}{n+1}$$
$$= \left\{\frac{(m+1)\,m}{1.2}\Delta + \frac{(m+1)\,m\,(m-1)}{1.2.3}\Delta^2, \&c.\right\} 0^n,$$

put P for m where $P^n = n$, then since

$$(x+P)^{n+1} = x^{n+1} + (n+1)(1+x)^n,$$

$$\frac{(B+1+P)^{n+1} - (B+1)^{n+1}}{n+1} = (B+2)^n = n + (B+1)^n = n + B_n;$$

also $\frac{(P+1)\,P(P-1)\ldots(P-r+2)}{1.2\ldots r} = (-1)^{r+1}.\frac{2}{r(r-1)(r-2)}\,(r>2)$.

Hence $n + B_n = \left\{\frac{(P+1)\,P}{1.2}\Delta + \frac{(P+1)\,P(P-1)}{1.2.3}\Delta^2, \&c.\right\} 0^n$

$$= \frac{3}{2}\Delta 0^n + 2\left\{\frac{\Delta^2}{1.2.3} - \frac{\Delta^3}{2.3.4} + \frac{\Delta^4}{3.4.5} - \&c.\right\} 0^n;$$

therefore

$$\left\{\frac{\Delta^2}{1.2.3} - \frac{\Delta^3}{2.3.4} + \frac{\Delta^4}{3.4.5} - \&c.\right\} 0^n = \frac{1}{4}(2B_n + 2n - 3) \ldots.. (15).$$

Ex. $(n=4)\ \frac{7}{6} - \frac{3}{2} + \frac{2}{5} = \frac{1}{4}\left(-\frac{1}{15} + 8 - 3\right) = \frac{37}{30}$.

10. Let

$$P^0 (1+x)^P = L^2 (1+x) = \frac{x}{1^2} - \frac{x^2}{2^2} + \frac{x^3}{3^2} - \&c. = \frac{(1+x)^{B+1} - 1}{B+1}$$

(obtained by integrating $(1+x)^B = 1 - \frac{x}{2} + \frac{x^2}{3} - \&c.$) For $(1+x)$ put ε^x, then

$$P^0 \varepsilon^{Px} = \frac{\varepsilon^{(B+1)x} - 1}{B+1} \text{ and } P^n = (B+1)^{n-1} = B_{n-1}\,(x>2).$$

Hence the equation $\Delta^0(1+\Delta)^P 0^n = P^n$ becomes $L^2(1+\Delta)^n 0^n = B_{n-1}$,

i.e. $\left(\frac{\Delta}{1^2} - \frac{\Delta^2}{2^2} + \frac{\Delta^3}{3^2} - \&c.\right) 0^n = B_{x-1}\,(x>2) \ldots\ldots (16).$

Again, let $(1+x)^P = \frac{1}{1^2} - \frac{x}{2^2} + \frac{x^2}{3^2} - \&c. = \frac{(1+x)^{B+1}-1}{(B+1)x}$,

then $$\varepsilon^{Px} = \frac{\varepsilon^{(B+1)x}-1}{B+1} \cdot \frac{1}{\varepsilon^x - 1} = \frac{1}{x} \cdot \frac{x}{\varepsilon^x - 1} \cdot \frac{\varepsilon^{(B+1)x}-1}{B+1}.$$

But $$\frac{x}{\varepsilon^x - 1} = \varepsilon^{Bx}.$$

Hence $$\varepsilon^{Px} = \frac{1}{x}\,\varepsilon^{B'x}\left(\frac{\varepsilon^{(x+1)x}-1}{B+1}\right),$$

the B' being accented in order that it may be isolated from the function of B with which it is connected, and thus receive its own separate development. Therefore

$$P^n = \frac{(B+1+B')^{n+1} - B'^{n+1}}{(n+1)(B+1)}$$

$$= \frac{(B+1)^n}{n+1} + B_1(B+1)^{n-1} + \frac{n}{2}B_2(B+1)^{n-2} + \ldots + \frac{n}{2}B_{n-1}(B+1) + B_n$$

$$= \frac{Bn}{n+1} + B_1B_{n-1} + \frac{n}{2}B_2B_{n-2}$$

$$+ \frac{n(n-1)}{2.3}B_3B_{n-3} + \ldots + \frac{n}{2}(B+1)B_{n-1} + B_n,$$

which therefore

$$= \Delta^0(1+\Delta)^P 0^n = -\left(\frac{\Delta}{2^2} - \frac{\Delta^2}{3^2} + \frac{\Delta^3}{4^2} - \&c.\right)0^n.$$

If n is odd, the left-hand side of this equation is reduced to $\frac{n-2}{4}B_{n-1}$ $(n>1)$. Hence

$$\left(\frac{\Delta}{2^2} - \frac{\Delta^2}{3^2} + \frac{\Delta^3}{4^2} - \&c.\right)0^n = -\frac{n-2}{4}B_{n-1}\ (n \text{ odd and} > 1) \ldots (17).$$

Ex. $(n=3)$, then $\frac{1}{4} - \frac{6}{9} + \frac{6}{16} = -\frac{1}{4}B_2 = -\frac{1}{24}$.

11. Let $(1+x)^Q = \frac{x}{1^3} - \frac{x^2}{2^3} + \frac{x^3}{3^3} - \&c.$, then differentiating

$$Q(1+x)^{Q-1} = \frac{1}{1^2} - \frac{x}{2^2} + \frac{x^2}{2^2} - \&c. = (1+x)^P;$$

therefore $Q(1+x)^Q = (1+x)^{P+1}$, $Q\varepsilon^{Qx} = \varepsilon^{(P+1)x}$, and $Q^n = (P+1)^{n-1}$.

But
$$\varepsilon^{Px} = \frac{1}{x}\left(\frac{\varepsilon^{(B'+B+1)x} - \varepsilon^{B'x}}{B+1}\right);$$

therefore
$$\varepsilon^{(P+1)x} = \frac{1}{x}\left(\frac{\varepsilon^{(B'+1B+1)x} - \varepsilon^{(B'+1)x}}{B+1}\right),$$

and
$$(P+1)^{n-1} = \frac{1}{n}\,\frac{\{(B'+1)+(B+1)\}^n - (B'+1)^n}{B+1}.$$

Hence $\Delta^0(1+\Delta)^Q 0^n \left\{= \left(\frac{\Delta}{1^3} - \frac{\Delta^2}{2^3} + \frac{\Delta^3}{3^3} - \&c.\right)0^n\right\} = Q^n = (P+1)^{n-1}$

$$= (B'+1)^{n-1} + \frac{n-1}{2}(B+1)\,(B'+1)^{n-2}$$

$$+ \frac{(n-1)(n-2)}{2.3}(B+1)^2(B'+1)^{n-3} + \ldots + (B+1)^{n-2}(B'+1) + \frac{(B+1)^{n-1}}{n}$$

$$= B_{n-1} + \frac{n-1}{4}B_{n-2} + \frac{(n-1)(n-2)}{2.3}B_2B_{n-3} + \ldots + \frac{1}{2}B_{n-2} + \frac{B_{n-1}}{n}$$

$$= \frac{n+1}{4}B_{n-2}\ (n \text{ even});$$

i.e. $L^3(1+\Delta)\,0^n$ which

$$= \left(\frac{\Delta}{1^3} - \frac{4^2}{2^3} + \frac{4^3}{3^3} - \&c.\right)0^n = \frac{n+1}{4}B_{n-2}\ (n \text{ even}) \ldots (18).$$

Ex. $\qquad (n=4),\ 1 - \frac{14}{8} + \frac{36}{27} - \frac{24}{64} = \frac{5}{4}B_2 = \frac{5}{24}.$

12. Differentiating $(1+x)^B = \frac{1}{x}\log(1+x)$, and multiplying by $1+x$, we have

$$B\,(1+x)^B = \frac{1}{x} - \left(\frac{1}{x} + \frac{1}{x^2}\right)\log(1+x),$$

repeating the process,

$$B^2(1+x)^B = -2\left(\frac{1}{x} + \frac{1}{x^2}\right) + \left(\frac{1}{x} + \frac{3}{x^2} + \frac{2}{x^3}\right)\log(1+x), \text{ similarly}$$

$$B^3\,(1+x)^B = 3\left(\frac{1}{x} + \frac{3}{x^2} + \frac{2}{x}\right) - \left(\frac{1}{x} + \frac{7}{x^2} + \frac{12}{x^3} + \frac{6}{x^4}\right)\log(1+x)$$

$$= 3\left(\frac{\Delta}{x} + \frac{\Delta^2}{2x^2} + \frac{\Delta^3}{3x^3}\right)0^3 - \left(\frac{\Delta}{x} + \frac{\Delta^2}{2x^2} + \frac{\Delta^3}{3x^3} + \frac{\Delta^4}{4x^4}\right)0^4\log(1+x).$$

Hence, generally

$$B^n (1+x)^B = (-1)^{n+1} \left\{ n \left(\frac{\Delta}{x} + \frac{\Delta^2}{2x^2}, \&c. \right) 0^n \right.$$

$$\left. - \left(\frac{\Delta}{x} + \frac{\Delta^2}{2x^2}, \&c. \right) 0^{n+1} \log(1+x) \right\} \text{..........} \quad \text{(IV)}.$$

For x put $\dfrac{-x}{1+x}$, then $B^n (1+x)^B$ becomes $B^n (1+x)^{-B}$, and $\log(1+x)$ becomes $-\log(1+x)$, therefore

$$B^n (1+x)^{-B} = (-1)^n n \left\{ \frac{1+x}{x} \Delta - \left(\frac{1+x}{x} \right)^2 \frac{\Delta^2}{2} + \left(\frac{1+x}{x} \right)^3 \frac{\Delta^3}{3} - \&c. \right\} 0^n$$

$$+ (-1)^{n+1} \log(1+x) \left\{ \frac{1+x}{x} \Delta - 1 \left(\frac{1+x}{x} \right)^2 \frac{\Delta^2}{2}, \&c. \right\} 0^{n+1}$$

$$\text{........................} \quad (19).$$

Hence $\qquad B^n (1+x)^B + B^n (1+x)^{-B}$

$$= (-1)^n n \left[\left(\frac{1+x}{x} - \frac{1}{x} \right) \Delta - \left\{ \left(\frac{1+x}{x} \right)^2 - \frac{1}{x^2} \right\} \frac{\Delta^2}{2}, \&c. \right] 0^n$$

$$+ (-1)^{n+1} \log(1+x) \left[\left(\frac{1+x}{x} + \frac{1}{x} \right) \Delta \right.$$

$$\left. - \left\{ \left(\frac{1+x}{x} \right)^2 - \frac{1}{x^2} \right\} \frac{\Delta^2}{2}, \&c. \right] 0^{n+1} \text{....................} \quad (20).$$

But $\qquad B^n (1+x)^B + B^n (1+x)^{-B}$

$$= 2B^n \left\{ 1 + \frac{B^2}{1.2} \log^2(1+x) + \frac{B^4}{1.2.3} \log^4(1+x), \&c. \right\}$$

$$= 2 \left\{ B_n + \frac{B_{n+2}}{1.2} \log^2(1+x) + \frac{B_{n+4}}{1.2.3.4} \log^4(1+x), \&c. \right\} = 0,$$

(n odd and > 1).

In this case, in the right-hand side of (20), the coefficient of $\log(1+x)$ must vanish, otherwise $\log(1+x)$ would always be expressed by a finite rational function of x, which is impossible.

Hence, putting $\dfrac{1}{x}$ for x, we have, from (20),

$$[\{(1+x) - x\} \Delta - \{(1+x)^2 + x^2\} \tfrac{1}{2}\Delta^2$$

$$+ \{(1+x)^3 - x^3\} \tfrac{1}{3}\Delta^3 - \&c.] \, 0^n = 0 \ (n \text{ odd}) \text{......} (21).$$

Similarly $\qquad B^n(1+x)^B - B^n(1+x)^{-B}$

$$=2\left\{B_n \log(1+x) + \frac{B_{n+2}}{1.2.3}\log^3(1+x), \text{ \&c.}\right\} = 0 \ (n \text{ even}),$$

and

$$[\{(1+x)+x\}\Delta - \{(1+x)^2 - x^2\}\tfrac{1}{2}\Delta^2$$
$$+\{(1+x)^3+x^3\}\tfrac{1}{3}\Delta^3 - \text{\&c.}]\,0^n = 0 \ (n \text{ even}) \ldots\ldots(22).$$

COR. From (20),

$$\log\{1+(1+x)\Delta\}\,0^n + \log(1-\Delta x)\,0^n = 0 \ (n \text{ odd}),$$

i.e. $\qquad \log\{1+\Delta - x(x+1)\Delta^2\}\,0^n = 0,$

or $\qquad \log(1+\Delta)\,(=0) + \log\left\{1 - x(x+1)\dfrac{\Delta^2}{1+\Delta}\right\}0^n = 0.$

Hence, since x is arbitrary,

$$\left(\frac{\Delta^2}{1+\Delta}\right)^m 0^n = 0 \ (n \text{ odd}) \ldots\ldots\ldots\ldots\ldots(23).$$

This result may be obtained also from (I), as follows:

$$f\left(\frac{\Delta^2}{1+\Delta}\right)0^n = U^0 f\frac{(U-1)^2}{U};$$

therefore

$$\left(\frac{\Delta^2}{1+\Delta}\right)^m = U^0\frac{(U-1)^{2m}}{U^m} = U^0(U-U^{-1})^{2m}$$

$$= U^0(U^{2m} + U^{-(2m)}) + \frac{2m}{1}(U^{2m-1} + U^{-(2m-1)}) + \text{\&c.}$$

$$= \{1+(-1)^n\}\left\{2m^n + \frac{2m}{1}(2m-1)^n, \text{ \&c.}\right\} = 0 \ (n \text{ odd}).$$

Hence $\qquad (\Delta^2 - \Delta^3 + \Delta^4, \text{ \&c.})\,0^n = 0,$

$$(\Delta^4 - 2\Delta^5 + 3\Delta^6 - \text{\&c.})\,0^n = 0, \text{ \&c., } (n \text{ odd}).$$

Ex. $n=7$; then

$$(\Delta^4 - 2\Delta^5 + 3\Delta^6 - 4\Delta^7)\,0^7 = 8400 - 2.16800 + 3.15120 - 4.5040 = 0.$$

13. Some remarkable results are obtained from (21) and (22) by substituting different representative quantities for x.

(1) For x put P, where $P^m = m$, then

$$\{\Delta - (2.2+3)\tfrac{1}{2}\Delta^2 + (3.2^2-2)\tfrac{1}{3}\Delta^3 - (4.2^3+5)\tfrac{1}{4}\Delta^4$$
$$+(5.2^4-4)\tfrac{1}{5}\Delta^4 - \text{\&c.}\}\,0^n = 0 \ (n \text{ odd}) \ldots\ldots(24),$$

and

$$\{(1+2)\,\Delta-(2.2-1)\,\tfrac{1}{2}\Delta^2+(3.2^2+4)\,\tfrac{1}{3}\Delta^3$$
$$-(4.2^3-3)\,\tfrac{1}{4}\Delta^4, \text{ \&c.}\}\,0^n=0 \quad (n \text{ even}) \ldots\ldots (25).$$

(2) In (21), for x put A, where $A_n=-\dfrac{2^{n+2}-2}{n+1}B_{n+1}$; then, after reduction,

$$\left(A_1\Delta+A_3\frac{\Delta^3}{3}+A_5\frac{\Delta^5}{5}+\ldots+A_n\frac{\Delta^n}{n}\right)0^n=0 \quad (n \text{ odd}),$$

i. e. $$\left\{(2^2-1)B_2\frac{\Delta}{1.2}+(2^4-1)B_4\frac{\Delta^3}{3.4}+(2^6-1)B_6\frac{\Delta^5}{5.6}, \text{ \&c.}\right\}0^{2m+1}=0$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (26).$$

(3) In (21), for x put B, then

$$(B_2\Delta^2+B_4\tfrac{1}{2}\Delta^4+B_6\tfrac{1}{3}\Delta^6, \text{ \&c.})\,0^{2m+1}=1 \ldots\ldots (27).$$

Ex. $(n=3)$, $\dfrac{1}{6}.126-\dfrac{1}{30}.\dfrac{8400}{2}+\dfrac{1}{42}.\dfrac{15120}{3}=1.$

(4) If in (21) we put $2B$ for x, then since

$$(2B+2)^n+(2B+1)^n=2\,(B+1)^n,$$

i.e. $$(2B+1)^n=(2-2^n)\,(B+1)^n=0 \quad (n \text{ odd}),$$

we shall have the same result as (27). But putting $2B$ for x in (22), we have (after reduction)

$$\{(2^2-1)\,B_2\Delta^2+(2^4-1)\,B_4\tfrac{1}{2}\Delta^4+(2^6-1)\,B_6\tfrac{1}{3}\Delta^6, \text{ \&c.}\}\,0^{2m}=1.$$

Ex. $(n=3)$, $\quad 31-390+360=1.$

(5) In (21) put Δ for x, applying the adjunct 0^m, then

$$(1+\Delta)\,0^m.\Delta0^n-(1+\Delta)^2\,0^m.\tfrac{1}{2}\,(\Delta^20^n)+(1+\Delta)^3\,0^m.\tfrac{1}{3}\,(\Delta^30^n)-\text{\&c.}$$
$$=\Delta0^m.\Delta0^n+\Delta^20^m.\tfrac{1}{2}\,(\Delta^20^n)+\Delta^30^m.\tfrac{1}{3}\,(\Delta^30^n), \text{ \&c.}$$

But $(1+\Delta)^r0^m=r^m$; therefore

$$1^{m-1}\Delta0^n-2^{m-1}\Delta^20^n+3^{m-1}\Delta^30^n-\text{\&c.}=\Delta0^m.\Delta0^n+\Delta^20^m.\tfrac{1}{2}(\Delta^20^n), \text{\&c.}$$

Let $m=n$,

$$(1-2^{n-1}\Delta^2+3^{n-1}\Delta^3-\text{\&c.})\,0^n$$
$$=1+\tfrac{1}{2}\,(\Delta^20^n)^2+\tfrac{1}{3}\,(\Delta^30^n)^3, \text{ \&c.} \quad (n \text{ odd}).$$

Ex. $n=3$, $\;1-4.6+9.6=1+\tfrac{1}{2}.36+\tfrac{1}{3}.36=31.$

If we make the substitution in (22), we get

$$(1^{m-1}\Delta - 2^{m-1}\Delta^2 + 3^{m-1}\Delta^3 - \&c.)\, 0^n$$
$$= -\{1 + \Delta^2 0^m . \tfrac{1}{2}(\Delta^2 0^n) + \Delta^3 0^m . \tfrac{1}{3}(\Delta^3 0^n), \&c.\} \quad (n \text{ even}),$$

so that, generally,

$$(1^{m-1}\Delta - 2^{m-1}\Delta^2, \&c.)\, 0^n = (-1)^{n+1}\{1 + \Delta^2 0^m . \tfrac{1}{2}(\Delta^2 0^n) + \&c.\}$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots(28).$$

(6) Let the symbolic zero be put for x in (21) and (22), then, applying the adjunct Δ^m, we have

$$\Delta^m (1+0) . \Delta 0^n - \Delta^m (1+0)^2 . \tfrac{1}{2}(\Delta^2 0^n) + \Delta^m (1+0)^3 . \tfrac{1}{3}(\Delta^3 0^n) - \&c.$$
$$= (-1)^{n+1}\{\Delta^m 0^n . \Delta 0^n + \Delta^m 0^2 . \tfrac{1}{2}(\Delta^2 0^n) + \&c.\}$$
$$= (-1)^{n+1}\left\{\Delta^m 0^n . \frac{\Delta^m 0^n}{m} + \Delta^m 0^{m+1} . \frac{\Delta^{m+1} 0^n}{m+1}, \&c.\right\}.$$

But (as stated in Art. 1)

$\Delta^m (1+0)^r$, where 1 is natural and 0 symbolic,

$= \Delta^m (1+0)^r$, where 1 is symbolic and 0 natural,

$= \Delta^m 1^r = \Delta^m (1+\Delta)\, 0^r$, from ($b$).

Hence, after reduction, the following singular results (not attainable perhaps by any other method):

$$\frac{1}{2}\left\{\frac{1}{m+1}\Delta^{m+1}0^{m+1} . \Delta^{m+1}0^n - \frac{1}{m+2}\Delta^{m+1}0^{m+2} . \Delta^{m+2}0^n \right.$$
$$\left. + \frac{1}{m+3}\Delta^{m+1}0^{m+3} . \Delta^{m+3}0^n - \&c.\right\}$$
$$= \frac{1}{m}\Delta^m 0^m . \Delta^m 0^n + \frac{1}{m+2}\Delta^m 0^{m+2} . \Delta^{m+2} 0^n$$
$$+ \frac{1}{m+4}\Delta^m 0^{m+4} . \Delta^{m+4}0^n - \&c. \quad (m+n \text{ odd})\ldots\ldots(29),$$

and $= -\left\{\dfrac{1}{m+1}\Delta^m 0^{m+1} . \Delta^{m+1}0^n + \dfrac{1}{m+3}\Delta^m 0^{m+3} . \Delta^{m+3} 0^n + \&c.\right\}$

$$(m+n \text{ even}) \ldots\ldots\ldots\ldots\ldots\ldots\ldots(30).$$

If for x in (21) and (22) we put symbolic r for x we should obtain general formulæ of which (29) and (30) would be particular cases.

14. **Expanding (IV), we have**

$$B^n\left\{1+Bx+\frac{B(B-1)}{1.2}x^2, \text{ \&c.}\right\}$$

$$=(-1)^{n+1}\left\{n\left(\frac{\Delta}{x}+\frac{\Delta^2}{2x^2}+\frac{\Delta^3}{3x^3}, \text{ \&c.}\right)0^n\right.$$

$$\left.-\left(x-\frac{x^2}{2}+\frac{x^3}{3}, \text{ \&c.}\right)\left(\frac{\Delta}{x}+\frac{\Delta^2}{2x^2}, \text{ \&c.}\right)0^{n+1}\right\}.$$

Hence, firstly, equating coefficients of $\frac{1}{x^m}$,

$$\frac{n}{m}\Delta^m 0^n=\left\{\frac{\Delta^{m+1}}{1\,(m+1)}-\frac{\Delta^{m+2}}{2\,(m+2)}, \text{ \&c.}\right\}0^{n+1}$$

$$=\frac{1}{m}\left\{\Delta^m\log(1+\Delta)\,0^{n+1}-\left(\frac{\Delta^{m+1}}{m+1}-\frac{\Delta^{m+2}}{m+2}, \text{ \&c.}\right)0^{n+1}\right\}.$$

But, from (4),

$$\Delta^m\log(1+\Delta)\,0^{n+1}=(n+1)\,\Delta^m 0^n;$$

therefore $$\Delta^m 0^n=\left(\frac{\Delta^{m+1}}{m+1}-\frac{\Delta^{m+2}}{m+2}, \text{ \&c.}\right)0^{n+1} \text{ .... } (\alpha).$$

Now let R be the representative of the reciprocal numbers, so that $R^c=\frac{1}{c}$, then

$$\Delta^m 0^n=\frac{R^{m+1}\Delta^{m+1}}{1+R\Delta}\,0^{n+1}=\frac{R\Delta}{1+R\Delta}\,(R\Delta)^m 0^{n+1}.$$

Hence $$f\Delta.0^n=\frac{R\Delta}{1+R\Delta}f(R\Delta)\,0^{n+1},$$

and $$\frac{1+\Delta}{\Delta}f\Delta.0^n=f(R\Delta)\,0^{n+1} \text{ .... (V)},$$

a formula prolific in results.

Let $f\Delta.0^n=\Delta^m 0^n$, then

$$(\Delta^m+\Delta^{m-1})\,0^n=\frac{\Delta^m}{m}.0^{n+1} \text{ .... (31)}.$$

If $f\Delta.0^n=0$, then

$$\frac{1}{\Delta}f\Delta.0^n=f(R\Delta)\,0^{n+1}.$$

Hence $\frac{1}{\Delta}\log(1+\Delta)\,0^n \left\{= \left(-\frac{\Delta}{2}+\frac{\Delta^2}{3}-\&c.\right)0^n\right\}$

$$=\left(\frac{\Delta}{1^2}-\frac{\Delta^2}{2^2}+\frac{\Delta^3}{3^2}-\&c.\right)0^{n+1}=B_n\;(n>1)\ldots\ldots(32).$$

By operating on (23), viz. $\left(\frac{\Delta^2}{1+\Delta}\right)^m 0^{2m-1}=0$, it is easy to obtain

$$(\Delta^2-\Delta^3+\Delta^5-\Delta^6+\Delta^8-\&c.)\,0^{2m-1}=0,$$

hence from (31)

$$(\Delta-\Delta^2+\Delta^4-\Delta^5+\Delta^7-\&c.)\,0^{2n-1}$$
$$=\left(\frac{\Delta^2}{2}-\frac{\Delta^3}{3}+\frac{\Delta^5}{5}-\frac{\Delta^6}{6}+\&c.\right)0^{2n}\ldots\ldots(33).$$

Secondly, equating coefficients of x^m, we have

$$\frac{B^{n+1}(B-1)(B-2)\ldots(B-m+1)}{1.2\ldots m}$$
$$=(-1)^{m+n}\left\{\frac{\Delta}{1\,(m+1)}-\frac{\Delta^2.}{2\,(m+2)},\ \&c.\right\}0^{n+1}.$$

Hence $\left(\frac{\Delta}{m+1}-\frac{\Delta^2}{m+2},\ \&c.\right)0^{n+1}$

$$=(-1)^{m+n+1}\frac{B^{n+1}(B-1)(B-2)\ldots(B-m+1)}{1.2\ldots(m-1)}\ldots(\text{VI}).$$

Ex. $(m=3,\ n=4)$, $\left(\frac{\Delta}{4}-\frac{\Delta^2}{5}+\frac{\Delta^3}{6}-\frac{\Delta^4}{7}+\frac{\Delta^5}{8}\right)0^5$

$$=\frac{B^5(B-1)(B-2)}{2}=\tfrac{1}{2}(B_7-3B_6+2B_5),$$

i.e. $\frac{1}{4}-\frac{30}{5}+\frac{150}{6}-\frac{240}{7}+\frac{120}{8}\left(=\frac{1}{4}-\frac{2}{7}\right)=\frac{3B_6}{2}=-\frac{3}{84}=-\frac{1}{28}.$

15. Numerous and interesting results may be obtained by the application of (c), viz.

$$\left(\frac{\Delta}{1+\Delta}\right)^m 0^n=(-1)^{m+n}\Delta^m 0^n.$$

(1) In $\frac{x}{m}-\frac{x^2}{m+1}+\frac{x^3}{m+2}-\&c.$

$$=\frac{1}{m}\cdot\frac{x}{1+x}+\frac{1}{m\,(m+1)}\cdot\left(\frac{x^2}{1+x}\right)+\frac{1.2}{m(m+1)(m+2)}\left(\frac{x}{1+x}\right)^3,\&c.$$

putting Δ for x and applying the adjunct 0^n, we have

$$\left\{\frac{1}{m}.\Delta - \frac{1}{m(m+1)}.\Delta^2 + \frac{1.2}{m(m+1)(m+2)}.\Delta^3 - \&c.\right\}0^n$$

$$= (-1)^{n+1}\left(\frac{\Delta}{m} - \frac{\Delta^2}{m+1} + \frac{\Delta^3}{m+2} - \&c.\right)0^n$$

$$= (-1)^m \frac{B^n(B-1\ldots(B-m+2)}{1.2\ldots(m-2)} \text{ from (VI)} \ldots\ldots\ldots\ldots\ldots (34).$$

Ex. $(m=4,\ n=3)$,

$$\left(\frac{\Delta}{4} - \frac{\Delta^2}{20} + \frac{\Delta^3}{60}\right)0^3 = \left(\frac{\Delta}{4} - \frac{\Delta^2}{5} + \frac{\Delta^3}{6}\right)0^3 = \tfrac{1}{2}(B_5 - 3B_4 + 2B_3),$$

i. e. $$\frac{1}{4} - \frac{3}{10} + \frac{1}{10} = \frac{1}{4} - \frac{6}{5} + 1 = -\frac{3}{2}B_4 = \frac{1}{20}.$$

(2) In $\log(1+x) = \dfrac{n}{1^2}\left(\dfrac{x}{1+x}\right) + \dfrac{n(n+1)}{1.2^2}\left(\dfrac{x}{1+x}\right)^2$

$$+ \frac{n(n+1)(n+2)}{1.2.3^2}\left(\frac{x}{1+x}\right)^3, \&c. - \left\{\frac{n-1}{1^2}x + \frac{(n-1)(n-2)}{1.2^2}x^2, \&c.\right\},$$

put Δ for x and apply the adjunct 0^m, then since

$$\log(1+\Delta)\,0^m = 0\ (m>1),$$

$$\left\{\frac{n}{1^2}\Delta - \frac{n(n+1)}{1.2^2}\Delta^2 + \&c.\right\}0^m$$

$$= (-1)^{m+1}\left\{\frac{n-1}{1^2}\Delta + \frac{(n-1)(n-2)}{1.2^2}\Delta^2 + \&c.\right\}0^m \ldots (35).$$

Ex. $(m=4,\ n=3)$,

$$\left(3\Delta - 3\Delta^2 + \frac{10}{3}\Delta^3 - \frac{15}{4}\Delta^4\right)0^4 = -\left(2\Delta + \frac{\Delta^2}{2}\right)0^4,$$

i.e. $$3 - 42 + 120 - 90 = -(2+7) = -9.$$

16. In all that has preceded (excepting (6), Art. 13) we have investigated the properties of the $\Delta^m 0^n$ numbers on the supposition that m alone varies, and n remains constant and positive. By making n to vary and also by giving to it negative values an immense variety of result may (by use of representative notation) be readily obtained.

(To be continued.)

ON TANGENTIAL COORDINATES.

By E. J. Routh, M.A., St. Peter's College, Cambridge.

Lemma. Let p be the distance of any point P from any straight line, q the distance of the polar reciprocal Q of the straight line from the polar reciprocal of the point; then $\dfrac{p}{q}=\dfrac{OP}{OQ}$. The reciprocals are to be taken with regard to any circle whose centre is O.

This proposition is proved in Salmon's *Conics*, and may be established geometrically as follows:

Let PM (fig. 11) be the perpendicular from P on any straight line $MH'K$, let the point Q be the polar reciprocal of the straight line, and $HK'N$ the polar line of P. Let QN be perpendicular to $HK'N$.

The triangles OHK', OKH' are evidently similar.

Therefore $$\frac{OH}{OK}=\frac{OH'}{OK'}.$$

But $$OH.OP=OK.OQ;$$

therefore $$\frac{OP}{OQ}=\frac{OH'}{OK'}.$$

Therefore $H'K'$ is parallel to PQ, and

$$\frac{OP}{OQ}=\frac{PH'}{QK'}.$$

But the triangles $QK'N$, $PH'M$ are similar; therefore

$$\frac{QN}{PM}=\frac{PH'}{QK'}=\frac{QO}{PO},$$

and the proposition is proved.

Let $A'B'C'$ (fig. 12) be any triangle, and let a point P be referred to it as a fundamental triangle by trilinear coordinates $\alpha\beta\gamma$. Reciprocate the figure with regard to any point O. Let A, B, C be the angular points of the new triangle, and let p, q, r be the tangential coordinates of the

polar line of P referred to ABC as a new fundamental triangle.

Then, by the lemma,

$$\alpha = p\frac{OP}{OA}, \quad \beta = q\frac{OP}{OB}, \quad \gamma = r\frac{OP}{OC}.$$

Hence, if the point P move above so that the trilinear equation

$$\phi(\alpha\beta\gamma) = 0$$

is satisfied, the line must move about so as always to touch the curve

$$\phi\left(p\frac{OP}{OA}, \quad q\frac{OP}{OB}, \quad r\frac{OP}{OC}\right) = 0.$$

But the function ϕ is homogeneous, hence OP will divide out of the equation.

Also, if O be the centre of the circle circumscribing the triangle ABC, we have

$$\phi(p, q, r) = 0.$$

Thus the very same equation is true both in trilinear and tangential coordinates.

If therefore we know the equation to any curve in trilinear coordinates, we may at once write down the equation to the corresponding curve in tangential coordinates.

Ex. 1. The polar line or tangent in the curve $\phi(\alpha, \beta, \gamma) = 0$ at the point $(\alpha\beta\gamma)$ is in trilinear coordinates

$$\frac{d\phi}{d\alpha}\alpha' + \frac{d\phi}{d\beta}\beta' + \frac{d\phi}{d\gamma}\gamma' = 0.$$

Therefore the pole or the point of contact in the curve $\phi(p, q, r)$ of the line (pqr) is in tangential coordinates

$$\frac{d\phi}{dp}p' + \frac{d\phi}{dq}q' + \frac{d\phi}{dr}r' = 0.$$

In order to transform the equations from trilinear to tangential coordinates readily, it is necessary to know the relations between the two triangles. Since OA', OB', OC' are respectively perpendicular to the sides BC, CA, AB, it follows that the angles A, B, C are respectively the supplements of the angles $B'OC'$, $C'OA'$, $A'OB'$. If O be the centre of the circle circumscribing ABC, then O is the centre of the circle inscribed in $A'B'C'$. Then we easily find

$$2A = \pi - A', \quad 2B = \pi - B', \quad 2C = \pi - C'.$$

The following examples of this method of finding equations in tangential coordinates have been selected because they admit of immediate verification by reference to Ferrers' *Trilinear Coordinates*, Chap. VII. It is evident however, that every formula in trilinear coordinates has its corresponding meaning in tangential coordinates.

Ex. 2. To find the centre of the conic $\phi(p, q, r)$ in tangential coordinates.

In trilinear coordinates the corresponding problem is, find the polar line of the centre of the inscribed circle.

The coordinates of this centre are $\alpha = \beta = \gamma$, hence the polar line is

$$\frac{d\phi}{d\alpha} + \frac{d\phi}{d\beta} + \frac{d\phi}{d\gamma} = 0.$$

Therefore the equation to the required centre is

$$\frac{d\phi}{dp} + \frac{d\phi}{dq} + \frac{d\phi}{dr} = 0.$$

Ex. 3. Find the asymptotes of $\phi(p, q, r) = 0$.*

The reciprocal problem is, find the points of contact of tangents drawn from the centre of the inscribed circle. These are evidently given by

$$\left.\begin{aligned} \phi(\alpha, \beta, \gamma) &= 0, \\ \frac{d\phi}{d\alpha} + \frac{d\phi}{d\beta} + \frac{d\phi}{d\gamma} &= 0. \end{aligned}\right\}$$

Hence the required asymptotes are given by

$$\left.\begin{aligned} \phi(p, q, r) &= 0, \\ \frac{d\phi}{dp} + \frac{d\phi}{dq} + \frac{d\phi}{dr} &= 0. \end{aligned}\right\}$$

Ex. 4. Find the condition $\phi(p, q, r) = 0$ is a parabola.

The reciprocal problem is, find the condition $\phi(\alpha, \beta, \gamma) = 0$ passes through the centre of the inscribed circle.

* Let $\phi(p, q, r) = 0$ be a curve of any degree, then if (p, q, r), $(p + dp$, &c.) be the coordinates of two successive tangents, we have

$$\frac{d\phi}{dp}\,dp + \frac{d\phi}{dq}\,dq + \frac{d\phi}{dr}\,dr = 0.$$

But two consecutive tangents meet at the point of contact. Hence, in the case of an asymptote, we have $dp = dq = dr$. Thus, the equations to find the asymptotes of $\phi(p, q, r) = 0$ are in all cases

$$\left.\begin{aligned} \frac{d\phi}{dp} + \frac{d\phi}{dq} + \frac{d\phi}{dr} &= 0, \\ \phi &= 0. \end{aligned}\right\}$$

The condition is clearly that the sum of the coefficients of $\phi=0$ should be equal to zero. We can also easily determine whether ϕ be an ellipse or hyperbola.

Ex. 5. In trilinear coordinates the inscribed conic and circle are

$$\left.\begin{array}{r}\sqrt{(L\alpha)}+\sqrt{(M\beta)}+\sqrt{(N\gamma)}=0\\ \cos\tfrac{1}{2}A'\,\sqrt{(\alpha)}+\cos\tfrac{1}{2}B'\,\sqrt{(\beta)}+\cos\tfrac{1}{2}C'\,\sqrt{(\gamma)}=0\end{array}\right\}.$$

Therefore in tangential coordinates the circumscribing conic and circle are

$$\left.\begin{array}{r}\sqrt{(Lp)}+\sqrt{(Mq)}+\sqrt{(Nr)}=0\\ \sin A\,\sqrt{(p)}+\sin B\,\sqrt{(q)}+\sin C\,\sqrt{(r)}=0\end{array}\right\}.$$

Ex. 6. In trilinear coordinates the circumscribing conic is

$$\frac{L}{\alpha}+\frac{M}{\beta}+\frac{N}{\gamma}=0.$$

Therefore in tangential coordinates the inscribed conic is

$$\frac{L}{p}+\frac{M}{q}+\frac{N}{r}=0.$$

In trilinear coordinates the circumscribing circle is

$$\frac{\sin A'}{\alpha}+\frac{\sin B'}{\beta}+\frac{\sin C'}{\gamma}=0.$$

Reciprocate this with regard to the centre of the circle as pole, then the equation in tangential coordinates to the inscribed circle is

$$\frac{OA\,\sin BOC}{p}+\frac{OB\sin AOC}{q}+\frac{OC\sin BOA}{r}=0.$$

But O being the centre of the circle inscribed in ABC, we have

$$OA\,\cos\frac{A}{2}=S-a,\quad \sin BOC=\cos\frac{B+C}{2};$$

therefore the equation becomes

$$\frac{S-a}{p}+\frac{S-b}{q}+\frac{S-c}{r}=0.$$

Ex. 7. By means of the formula for the distance between two points given in Ferrers' *Trilinear Coordinates*, p. 6, we can write down the general equation to a conic having its focus at the given point $\alpha_0\beta_0\gamma_0$.

Let $l\alpha + m\beta + n\gamma = 0$ be the directrix, then the conic is

$$a(\beta-\beta_0)(\gamma-\gamma_0)+b(\gamma-\gamma_0)(\alpha-\alpha_0)+c(\alpha-\alpha_0)(\beta-\beta_0) = (l\alpha+m\beta+n\gamma)^2.$$

If $\alpha_0\beta_0\gamma_0$ be the centre of the inscribed circle, this becomes

$$a(S-a)\alpha^2+\ldots-2(S-b)(S-c)\beta\gamma-\ldots=(l\alpha+m\beta+n\gamma)^2,$$

or $$a(S-a)(\alpha-\beta)(\alpha-\gamma)+\ldots=(l\alpha+m\beta+n\gamma)^2.$$

Hence, in tangential coordinates, the general equation to a circle is

$$\sin^2 A\,(p-q)(p-r)+\sin^2 B\,(q-p)(q-r) + \sin^2 C\,(r-p)(r-q)=(l'\alpha+m'\beta+n'\gamma)^2,$$

where $l'\alpha+m'\beta+n'\gamma=0$ is the equation to the centre.

Another view of Tangential Coordinates.

Let the position of a point P be defined as the centre of gravity of three weights (l, m, n) placed at the angular points A, B, C of the fundamental triangle. Draw any straight line through P, and let (p, q, r) be its distances from the angular points A, B, C. Then taking moments about this line, we have

$$lp+mq+nr=0.$$

If $l+m+n=0$, the three weights make a couple, and the point P is at infinity. The equation to a point is said to be "prepared" when $l+m+n=1$.

Ex. Given two points P and Q, whose prepared equations in tangential coordinates, are

$$\left.\begin{aligned} lp+mq+nr&=0\\ l'p+m'q+n'r&=0\end{aligned}\right\},$$

find a point R which divides the distance PQ in the ratio $h:k$.

The point R is evidently the centre of gravity of the weights $\frac{1}{h}$, $\frac{1}{k}$ placed at P and Q. But a weight unity at P is equivalent to weights (l, m, n) at A, B, C. So also unity at Q is equivalent to (l', m', n') at A, B, C. Hence, R is the centre of gravity of the weights

$$\frac{l}{h}+\frac{l'}{k},\ \frac{m}{h}+\frac{m'}{k},\ \frac{n}{h}+\frac{n'}{k} \text{ at } ABC.$$

Hence the equation to R, is

$$\left(\frac{l}{h}+\frac{l'}{k}\right)p+\left(\frac{m}{h}+\frac{m'}{k}\right)q+\left(\frac{n}{h}+\frac{n'}{k}\right)r=0.$$

Let the equation to a curve in tangential coordinates, be

$$\phi\,(pqr)=Ap^2+Bq^2+Cr^2+2A'qr+2'B'rp+2C'pq=0.$$

Then, since $2pq=(p+q)^2-p^2-q^2$, the equation may be written

$$(A-B'-C')\,p^2+(B-C'-A')\,q^2+(C-A'-B')\,r^2$$
$$+4A'\left(\frac{q+r}{2}\right)^2+4B'\left(\frac{r+p}{2}\right)^2+4C'\left(\frac{p+q}{2}\right)^2=0.$$

Let three weights $A-B'-C'$, $B-C'-A'$, $C-A'-B'$ be placed at the angular points of the fundamental triangle, and three other weights $4A'$, $4B'$, $4C'$ at the middle points of the sides. Then the straight line (pqr) moves about, so that the moment of inertia of these six weights about it is constant, and = zero.

By propositions proved in Rigid Dynamics, we know that if a straight line move in one plane so that the moment of inertia about it is constant and equal to Q, then it envelopes a conic whose centre is the centre of gravity of the body, and whose principal diameters are the principal axes of the six weights at their centre of gravity, and whose cartesian equation referred to these axes, is

$$\frac{X^2}{Q-J}+\frac{Y^2}{Q-I}=1,$$

where I and J are the squares of the radii of gyration about the axes of X and Y. In our case $Q=0$.

In finding the centre we may substitute for the weights placed at the middle points of the sides, their halves placed at the angular points of the triangle. Hence, the centre of the conic is the centre of gravity of the three weights

$$A+B'+C',\ B+C'+A',\ C+A'+B',$$

placed at the angular points. Its tangential equation is therefore

$$(A+B'+C')p+(B+C'+A')\,q+(C+A'+B')\,r=0.$$

The centre of gravity of the weights being taken as origin, and any two straight lines at right angles as axes, we can easily find the values of Σmx^2, Σmy^2, and Σmxy. Let θ be the angle either principal diameter of the conic makes with the axis of x, and let $-I$ and $-J$ be the squares of the semi-axes of the conic. Then the directions of the axes are given by

$$\tan 2\theta = \frac{2\Sigma mxy}{\Sigma m(x^2 - y^2)},$$

and the lengths of the axes by

$$\left.\begin{aligned} I + J &= -\Sigma m(x^2 + y^2) \\ IJ &= (\Sigma mx^2)(\Sigma my^2) - (\Sigma mxy)^2 \end{aligned}\right\}.$$

The foci of the conic possess the property that the moments of inertia about all straight lines through them are the same. Let $\bar{\alpha}$, $\bar{\beta}$, $\bar{\gamma}$ be the trilinear coordinates of the centre, and α, β, γ the trilinear coordinates of one focus. Let p_1, p_2, p_3 be the perpendiculars from the angular points on the opposite sides. Then the moments of inertia of the six weights about the three sides are Ap_1^2, Bp_2^2, Cp_3^2. And

$$Ap_1^2 - S\bar{\alpha}^2 + S(\alpha - \bar{\alpha})^2$$

is equal to the moment of inertia about a line through the focus parallel to the side α, where S = sum of the six weights. Hence, we have the following equations to find the foci

$$\frac{Ap_1^2}{S} + \alpha(\alpha - 2\bar{\alpha}) = \frac{Bp_2^2}{S} + \beta(\beta - 2\bar{\beta}) = \frac{Cp_3^2}{S} + \gamma(\gamma - 2\bar{\gamma}),$$

$$a\alpha + b\beta + c\gamma = \Delta.$$

The conic $\phi(pqr) = Q$ is clearly confocal with the conic $\phi(pqr) = 0$. To make the equation homogeneous, we must write it in the form

$$\phi(pqr) = Q'\{a^2(p-q)(p-r) + b^2(q-p)(q-r) + c^2(r-p)(r-q)\}.$$

By making the foci coincide with the centre, we may easily deduce the condition that the general equation belongs to a circle. In this case we have $\alpha = \bar{\alpha}$, $\beta = \bar{\beta}$, $\gamma = \bar{\gamma}$. Hence

$$\frac{Ap_1^2}{S} - \bar{\alpha}^2 = \frac{Bp_2^2}{S} - \bar{\beta}^2 = \frac{Cp_3^2}{S} - \bar{\gamma}^2.$$

The centre being the centre of gravity of the three weights $A+B'+C'$, $B+C'+A'$, $C+A'+B'$ placed at the angular points, we have

$$S\bar{\alpha}=(A+B'+C')\,p,$$

and similar forms for $\bar{\beta}$ and $\bar{\gamma}$. Substituting for $\bar{\alpha}, \bar{\beta}, \bar{\gamma}$, we get

$$\{AS-(A+B'+C')^2\}\,p_1^2=\text{similar expressions.}$$

But since $ap_1=bp_2=cp_3=$ twice the area of the triangle, this reduces to

$$\frac{A(2A'+B+C)-(B'+C')^2}{a^2}=\frac{B\{2B'+C+A\}-\{C'+A'\}^2}{b^2}$$

$$=\frac{C\{2C'+A+B\}-(A'+B')^2}{c^2},$$

which is the required condition that the conic is a circle.

We may also find the condition that the general equation belongs to a rectangular hyperbola. The condition is obviously $I+J=0$. Let G be the centre of the conic, whose coordinates have just been found, and let $\bar{r}$ be the distance of G from any one (m) of the six weights, then the condition becomes

$$\Sigma m\bar{r}^2=0.$$

By a known theorem in Statics, if r be the distance of the weight m from an angular point A, this is the same as

$$\Sigma mr^2-S(GA)^2=0.$$

After substitution, this becomes

$$\{(B+A')c^2+(C+A')b^2-A'a^2\}\{A+B+C+2A'+2B'+2C'\}$$
$$=(B+C'+A')^2c^2+(C+A'+B')^2b^2$$
$$+(B+C'+A')(C+A'+B')(b^2+c^2-a^2).$$

By examining the coefficient of a^2 we easily see that the required condition that the conic is a rectangular hyperbola is

$$a^2\{A'(A'+A+B'+C')-(B+C')(B'+C)\}$$
$$+b^2\{B'(B'+B+C'+A')-(C+A')(C'+A)\}$$
$$+c^2\{C'(C'+C+A'+B')-(A+B')(A'+B)\}=0.$$

ANALYTICAL METRICS.

By W. K. CLIFFORD.

(Continued from p. 21).

V. Planes and Points in Space.

Expressions Considered.

17. IN Geometry of Three Dimensions the absolute is an imaginary circle in which the plane at infinity cuts any sphere. The condition that a plane $A=0$ may touch this circle, is of the second order in the coefficients of A, and may be written $\phi(A)=0$. This being so, we have, as before,

$$\phi(\lambda A+\mu B)\equiv\lambda^2\phi(A)+2\lambda\mu.\psi(A, B)+\mu^2.\phi(B),$$

where the function $\psi(A, B)$ is of the first order in the coefficients of each of the two planes, and vanishes only when they are at right angles.

The condition that a point $a=0$ may be at infinity is of the first order in the coordinates or coefficients of the point, and may be written $a\infty=0$.

The condition that four planes $A, B, C, D=0$ may meet in a point, is that the determinant formed with their coordinates or coefficients, that is to say, their Jacobian, shall vanish. This I express by the equation

$$J(ABCD)=0.$$

The condition that three planes $A, B, C=0$ may meet at infinity, or, which is the same thing, may be parallel to the same line, is accordingly

$$J(ABC\infty)=0.$$

18. The form of the absolute ϕ is found exactly as in plane geometry; I will therefore anticipate a formula, and give it here. We have generally,

$$\phi(lx+my+nz+sw)\equiv l^2\phi(x)+\ldots+2mn\psi(y, z)+\ldots.$$

Now by means of the formula

$$\cos(A, B)=\frac{\psi(A, B)}{\sqrt{\{\phi(A).\phi(B)\}}},$$

this becomes, in quadriplanar coordinates,

$$\phi\,(lx+my+nz+sw) \equiv l^2+m^2+s^2+w^2-2mn\cos(y,\,z)-\ldots,$$

and in tetrahedral coordinates

$$\phi\,(lx+my+nz+sw) \equiv l^2\alpha^2+\ldots-2mn.\beta\gamma.\cos(y,\,z)-\ldots,$$

in both of which $\cos(y, z)$ means the cosine of the internal angle between the planes $y=0$, $z=0$ of the fundamental tetrahedron, and α, β, γ, δ are the areas of the faces. The equation to the plane at infinity is then found to be, in the quadriplanar system,

$$Px+Qy+Rz+Sw=0,$$

where $$P^2 = \begin{vmatrix} 1, & -\cos(y,z), & -\cos(y,w) \\ -\cos(y,z), & 1, & -\cos(z,w) \\ -\cos(y,w), & -\cos(z,w), & 1 \end{vmatrix}.$$

It is convenient to call P the *sine* of the solid angle yzw (M. Paul Serret). If we write A, B, C, D for the solid angles of the tetrahedron, then

$$\frac{\sin A}{\alpha}=\frac{\sin B}{\beta}=\frac{\sin C}{\gamma}=\frac{\sin D}{\delta},$$

and the equation to the plane at infinity, in tetrahedral coordinates, is

$$\beta\gamma\delta\sin A\,(x+y+z+w)=0.$$

Fundamental Propositions.

19. The following propositions may be proved by Cartesian coordinates.

A. If a plane touch the imaginary circle at infinity,

(*a*) Every area measured on the plane is zero.

(*b*) Every perpendicular distance from it is infinite.

(*c*) Of every angle *on* the plane, the sine is zero and the cosine unity.

(*d*) Of every angle made with the plane, the sine and cosine are infinite.

B. If a straight line meet the imaginary circle at infinity,

(*a*) Every length measured on the line is zero.

(*b*) Every perpendicular distance from it is infinite.

(*c*) Of the angle between any two planes through it, the sine is zero and the cosine unity.

(*d*) Of every angle made with it, the sine and cosine are infinite.

C. If a point be at infinity,

(*a*) Its perpendicular distance from any plane or straight line not passing through it is infinite.

(*b*) The volume of the tetrahedron which it forms with any other three points not in the same plane with it is infinite.

(*c*) The area of the triangle which it forms with any other two points, not such that the plane of the triangle touches the absolute (see prop. *A*, *a*), is infinite.

(*d*) Its distance from any other point, not at infinity, is infinite.

Formulæ of Adaptation.

20. These are to be deduced from the propositions just stated precisely in the same way as the corresponding formulæ in Plane Geometry were proved. I postpone for the present the consideration of formulæ relating to straight lines.

The volume contained by four points a, b, c, d is

$$\frac{J(abcd)}{a\infty \,.\, b\infty \,.\, c\infty \,.\, d\infty}.$$

If the points are given as the intersections of four planes, A, B, C, D, the expression becomes

$$\frac{\{J(ABCD)\}^3}{J(BCD\infty).J(CDA\infty).J(DAB\infty).J(ABC\infty)}.$$

The area contained by three points a, b, c is

$$\frac{\sqrt{\phi(abc)}}{a\infty \,.\, b\infty \,.\, c\infty},$$

where $\phi(abc)$ means that we are to write down the equation of the plane through a, b, c, and then form the condition that it may touch the imaginary circle at infinity. If the three points are given as intersections of the plane A with the planes B, C, D, the formula becomes

$$\frac{\sqrt{\phi A}.\{J(ABCD)\}^2}{J(ACD\infty).J(ADB\infty).J(ABC\infty)},$$

which is the area determined on the plane A by the planes B, C, D.

The perpendicular from the point a on the plane A is $\dfrac{aA}{\sqrt{\phi A . a\infty}}$. Here aA is used for the result of substituting the coordinates of a in the equation of A, or *vice versâ*.

If θ be the angle between two planes A and B, then

$$\cos\theta = \frac{\psi(A, B)}{\sqrt{(\phi A . \phi B)}}, \quad \sin^2\theta = \frac{\phi A . \phi B - \{\psi(A, B)\}^2}{\phi A . \phi B}.$$

It will be proved that the sine of the solid angle contained by three planes is

$$\frac{J(ABC\infty)}{\sqrt{\{\phi A . \phi B . \phi C\}}}.$$

Theorems.

21. I use Professor Sylvester's umbral notation, which, whenever determinants have to be employed, is not only convenient, but essential.

In this notation, the determinant

$$\begin{vmatrix} \psi(AD), & \psi(BD), & \psi(CD) \\ \psi(AE), & \psi(BE), & \psi(CE) \\ \psi(AF), & \psi(BF), & \psi(CF) \end{vmatrix}$$

is written $$\psi \left| \begin{matrix} ABC \\ DEF \end{matrix} \right. .$$

It will be easy now to interpret the notation in other cases. For instance, the definition of $\sin(y, z, w)$, in Art. (18), may be written

$$\sin^2(y, z, w) \equiv \cos \left| \begin{matrix} yzw \\ yzw \end{matrix} \right. .$$

I consider now the series of determinants

$$\psi \left| \begin{matrix} ABCD \\ EFGH \end{matrix} \right. , \quad \psi \left| \begin{matrix} ABC \\ DEF \end{matrix} \right. , \quad \psi \left| \begin{matrix} AB \\ CD \end{matrix} \right. .$$

Suppose we wanted to find the condition that it might be possible to draw a plane

$$\lambda A + \mu B + \nu C + \sigma D = 0,$$

which should be at once perpendicular to each of the planes E, F, G, H. We should write down four equations like

$$\lambda\psi(A, E) + \mu\psi(B, E) + \nu\psi(C, E) + \sigma\psi(D, E) = 0,$$

and then eliminate $\lambda\mu\nu\sigma$ between these equations. Thus, we should arrive at the condition $\psi\left|\begin{matrix}ABCD\\EFGH\end{matrix}\right.=0$. First, suppose that A, B, C, D do not meet in a point. Then the equation of any plane whatever can be put into the form

$$\lambda A+\mu B+\nu C+\sigma D=0.$$

Now, the plane at infinity is perpendicular to all other planes. In this case, therefore, the condition $\psi\left|\begin{matrix}ABCD\\EFGH\end{matrix}\right.=0$ is satisfied. Next let A, B, C, D meet in a point. Then there is an *identity*

$$\lambda A+\mu B+\nu C+\sigma D\equiv 0,$$

which gives rise to four other identities like

$$\lambda\psi(AE)+\mu\psi(BE)+\nu\psi(CE)+\sigma\psi(DE)\equiv 0,$$

so that in this case also $\psi\left|\begin{matrix}ABCD\\EFGH\end{matrix}\right.=0$.

Hence, we have always, identically,

$$\psi\left|\begin{matrix}ABCD\\EFGH\end{matrix}\right.\equiv 0 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\; (1).$$

In the same way we see, that $\psi\left|\begin{matrix}ABC\\DEF\end{matrix}\right.=0$, expresses the condition that a plane $\lambda A+\mu B+\nu C=0$ may be so drawn as to be perpendicular at once to each of the planes D, E, F. Now unless three planes are parallel to the same line, the only plane which is perpendicular to all three of them is the plane at infinity. If therefore the condition is satisfied, either A, B, C meet at infinity, or D, E, F meet at infinity. Thus we have

$$\psi\left|\begin{matrix}ABC\\DEF\end{matrix}\right.\equiv J(ABC\infty).J(DEF\infty)\;\ldots\ldots\; (2).$$

The condition $\psi\left|\begin{matrix}AB\\CD\end{matrix}\right.=0$ will be satisfied if we can draw a plane through the line (A, B) perpendicular to the two planes C, D. It is easy to see that this can only be the case when the line (A, B) is perpendicular to the line (C, D). This result may be expressed in the form

$$\psi\left|\begin{matrix}AB\\CD\end{matrix}\right.\equiv\psi(AB,\, CD)\;\ldots\ldots\ldots\ldots\ldots\; (3).$$

To interpret the theorems (1) and (2), I observe that to multiply any single row or column of a determinant by a

certain quantity, is to multiply the whole determinant by that quantity. Thus we have

$$\cos\begin{vmatrix} ABCD \\ EFGH \end{vmatrix} = \frac{\psi\begin{vmatrix} ABCD \\ EFGH \end{vmatrix}}{\sqrt{\{\phi A.\phi B.\phi C.\phi D.\phi E.\phi F.\phi G.\phi H\}}} = 0,$$

and $$\cos\begin{vmatrix} ABC \\ DEF \end{vmatrix} = \frac{\psi\begin{vmatrix} ABC \\ DEF \end{vmatrix}}{\sqrt{\{\phi A.\phi B.\phi C.\phi D.\phi E.\phi F\}}}$$

$$= \frac{J(ABC\infty)}{\sqrt{\{\phi A.\phi B.\phi C\}}}\cdot\frac{J(DEF\infty)}{\sqrt{\{\phi D.\phi E.\phi F\}}}.$$

If now we make A, B, C identical with D, E, F respectively, we get

$$\frac{\{J(ABC\infty)\}^2}{\phi A.\phi B.\phi C} = \cos\begin{vmatrix} ABC \\ ABC \end{vmatrix} = \sin^2(A, B, C) \text{ by definition;}$$

and so $$\cos\begin{vmatrix} ABC \\ DEF \end{vmatrix} = \sin(A, B, C).\sin(D, E, F).$$

The theorems (1) and (2) will be found to embody a great number of results.

22. I now prove certain known theorems of determinants, which are useful in this subject.

Consider five planes A, B, C, D, E, whose equations are

$$l_1x + m_1y + n_1z + s_1w = 0,$$

$$\dots\dots\dots\dots\dots\dots\dots\dots$$

$$l_5x + m_5y + n_5z + s_5w = 0.$$

We know that, since each one can be expressed in terms of the other four, there must exist some identical relation

$$PA + QB + RC + SD + TE = 0.$$

And because this is true for all values of $x_1y_1z_1w_1$, we must have

$$Pl_1 + Ql_2 + Rl_3 + Sl_4 + Tl_5 = 0,$$
$$\&c. \qquad \&c.$$

From these five equations we can eliminate $PQRST$, and we find

$$A.J(BCDE) + B.J(CDEA) + C.J(DEAB) + D.J(EABC) + E.J(ABCD) \equiv 0 \;\dots\dots\dots\dots\; (4),$$

the identical relation in question.

Now if we substitute in A the coordinates of the intersection of F, G, H, the result is clearly $J(AFGH)$. But the equation (4) is true for all values of the variables; we may therefore substitute in it the coordinates of the point (F, G, H). In this way we obtain

$$\begin{aligned} &J(AFGH).J(BCDE)+J(BFGH).J(CDEA)\\ &+J(CFGH).J(DEAB)+J(DFGH).J(EABC)\\ &+J(EFGH).J(ABCD)\equiv 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5). \end{aligned}$$

Making H identical with E, and transposing, we have

$$\begin{aligned} &J(ABCE).J(DFGE)\equiv J(AFGE).J(BCDE)\\ &\quad +J(BFGE).J(CADE)+J(CFGE).J(ABDE)\ldots.(6). \end{aligned}$$

Write ∞ for E, and use theorem (2); thus

$$\cos\left|\begin{matrix}ABC\\DFG\end{matrix}\right.\equiv\cos\left|\begin{matrix}AFG\\BCD\end{matrix}\right.+\cos\left|\begin{matrix}BFG\\CAD\end{matrix}\right.+\cos\left|\begin{matrix}CFG\\ABD\end{matrix}\right.\ldots(7).$$

Again, let E be the plane $w=0$ of the fundamental tetrahedron; and write (1, 2, 3, 4, 5, 6), for the traces on this plane of the planes (A, B, C, D, F, G) respectively; then we get the theorem in plane geometry

$$\begin{aligned} J(123).J(456)\equiv J(156).J(234)+J(164).J(235)&\\ +J(145).J(236)&. \end{aligned}$$

If we make the lines 1, 2 identical, and write ∞ for each, this becomes the theorem referred to in Art. (6).

Spheres.

23. *To find the equation to the sphere whose diameter is the line joining the point (A, B, C), to the point (D, E, F).*

The sphere may be defined as the locus of the foot of a perpendicular from the point (A, B, C) on a variable plane passing through the point (D, E, F). Now the equations to a line through (A, B, C) perpendicular to a plane L are

$$\frac{A}{\psi(A, L)}=\frac{B}{\psi(B, L)}=\frac{C}{\psi(C, L)}=\frac{1}{k}, \text{ suppose.}$$

Let now $L\equiv\lambda D+\mu E+\nu F$; then at the foot of the perpendicular,

$$\begin{aligned} \lambda\psi(A, D)+\mu\psi(A, E)+\nu\psi(A, F)-kA&=0,\\ \lambda\psi(B, D)+\mu\psi(B, E)+\nu\psi(B, F)-kB&=0,\\ \lambda\psi(C, D)+\mu\psi(C, E)+\nu\psi(C, F)-kC&=0,\\ \lambda D\quad+\quad\mu E\quad+\quad\nu F\quad&=0, \end{aligned}$$

from these equations we can eliminate λ, μ, ν, k, and we get for the equation of the sphere

$$\begin{vmatrix} \psi AD, & \psi AE, & \psi AF, & A \\ \psi BD, & \psi BE, & \psi BF, & B \\ \psi CD, & \psi CE, & \psi CF, & C \\ D & E & F & 0 \end{vmatrix} = 0.$$

Call this $(\psi)\begin{vmatrix} DEF0 \\ ABC0 \end{vmatrix} = 0$; then since theorem (6) is a general theorem of determinants, we have,

$$(\psi)\begin{vmatrix} ABC0 \\ DEF0 \end{vmatrix} \equiv (\psi)\begin{vmatrix} AEF0 \\ BCD0 \end{vmatrix} + (\psi)\begin{vmatrix} AFD0 \\ BCE0 \end{vmatrix} + (\psi)\begin{vmatrix} ADE0 \\ BCF0 \end{vmatrix};$$

.... (8),

these four spheres have therefore a common radical axis. Whence this geometrical theorem:—

If the faces of a tetrahedron A, B, C, D are met by a straight line in the points a, b, c, d; then the spheres whose diameters are Aa, Bb, Cc, Dd have a common radical axis.

It will be observed that the *faces* of the tetrahedron are A, D, E, $F = 0$, and that the *straight line* is the intersection of the planes $B = 0$, $C = 0$. The relation (8) is very easily verified.

(To be continued.)

ON SOME SPECIAL FORMS OF CONICS.

By C. Taylor, M.A., St. John's College, Cambridge.

When, in the third volume of the *Messenger of Mathematics*, I described as erroneous the notion that two conjugate points might be regarded as the limit of a conic, I had arrived at that conclusion, whether from *à priori* argument or by direct demonstration, without considering the statements of some eminent mathematicians to whom I shall presently allude. My opinion still remaining unchanged, I shall first investigate what seems to me the correct result, and afterwards proceed to examine some arguments of an opposite tendency.

1. Let an ellipse be regarded as the locus of a point P (fig. 13), which moves subject to the condition that $FP + F'P$ is constant, F, F' being fixed points. Now, when the minor

axis vanishes, the constant becomes equal to FF', and the limit of the curve is the straight line FF', which may therefore be regarded as an ellipse whose foci are F, F', and whose minor axis is equal to zero.

The same result follows from the equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, which assumes an indeterminate form when b vanishes, but is easily shown to be satisfied by zero values of y, simultaneously with an indeterminate value of x between the limits $\pm\frac{1}{2}FF'$. Moreover, it would involve an incongruity to suppose that any Cartesian equation of the second degree could in a limiting or any other form represent two conjugate points, or, in other words, that by the existence of some relation between its coefficients, an equation could be raised from the second to the fourth degree, for a point (a, b) is represented by an equation of the form $(x-a)^2+(y-b)^2=0$, and consequently the equation of *two* points is of the fourth degree.

Tangential Coordinates.

2. Dr. Salmon has shown in his *Conic Sections* (Fourth Edition, p. 248), that the envelope of a straight line whose perpendicular distances from three fixed points are connected by any equation of the second degree (in addition to the *identical* relation which subsists between them) is a conic, that is, a curve represented by a Cartesian or trilinear equation of the second degree. It is hence evident that the original tangential equation cannot in virtue of any relation between its coefficients become equivalent to a Cartesian equation of the fourth degree, and cannot therefore when it resolves itself into the form

$$(l\lambda+m\mu+n\nu)(l'\lambda+m'\mu+n'\nu)=0,$$

represent the two points obtained by equating the factors separately to zero.

In order to investigate the real nature of this limiting case, it will be well to consider the tangential equation in the form $\mu\nu=b^2$, to which it may be reduced by appropriate assumptions. The equation in this form represents the fact that the rectangle under the perpendiculars (μ, ν) from two fixed points (F, F') upon any tangent to the ellipse, is constant, and equal to the square on half the minor axis.

If $c=\frac{1}{2}FF'$, the straight line which moves subject to the

given condition is represented by the Cartesian equation, (fig. 13)

$$x\cos\alpha + y\sin\alpha = \sqrt{(b^2 + c^2\cos^2\alpha)}\ldots\ldots\ldots\ldots\text{(i)},$$

and the problem for solution is to determine the locus of ultimate intersections of the straight lines represented by this equation.

Differentiate (i) with respect to α, thus

$$-x\sin\alpha + y\cos\alpha = \frac{c^2\sin\alpha\cos\alpha}{\sqrt{(b^2 + c^2\cos^2\alpha)}}\ldots\ldots\ldots\text{(ii)}.$$

Solving (i) and (ii) as simultaneous equations, we obtain for the coordinates of the point of intersection of two consecutive positions of the line (i),

$$x = \frac{(b^2 + c^2)\cos\alpha}{\sqrt{(b^2 + c^2\cos^2\alpha)}};\quad y = \frac{b^2\sin\alpha}{\sqrt{(b^2 + c^2\cos^2\alpha)}}.$$

Now, in the equation $\mu\nu = b^2$, let b be indefinitely diminished, so that $y = 0$, and $x = \pm c$, *except when* $\cos\alpha$ *also vanishes.* In this case it may be shown that y still vanishes, whilst x assumes the form

$$\left\{\frac{b^2 + c^2}{\sqrt{\left(\frac{b^2}{\cos^2\alpha} + c^2\right)}} = \right\}\frac{c^2}{\sqrt{\left(\frac{0}{0} + c^2\right)}},$$

and is therefore indeterminate between the limits $\pm c$. We are thus led to the conclusion that the envelope of the line whose coordinates are connected by the relation $\mu\nu = 0$, is the straight line which joins the points $\mu = 0$, $\nu = 0$.

If $\mu\nu = -b^2$, the envelope, when $b = 0$, is the infinite straight line drawn through F, F', with the exception of the part intermediate to those points.

That the expression $lt.\dfrac{b}{\cos\alpha}$, in the expression for x, is really indeterminate, may be further shown as follows (fig. 13):

Take any position of the line (i), and let it cut FF' in T. Draw the perpendiculars FZ, $F'Z'$, and denote CT by δ.

Then $$\cos^2\alpha = \frac{FZ}{CT - CF}\cdot\frac{F'Z'}{CT + CF'}$$
$$= \frac{b^2}{\delta^2 - c^2};$$

therefore $$x = \left\{\frac{b^2 + c^2}{\sqrt{\left(\frac{b^2}{\cos^2\alpha} + c^2\right)}} = \right\} = \frac{b^2 + c^2}{\delta},$$

a result at which we might have arrived at once by assuming as a geometrical property of the ellipse that $CN.CT = CA^2$, with the usual notation. It is hence evident that we may regard PT as assuming its limiting position by the evanescence of FZ *and* $F'Z'$, whilst CT has some value intermediate to c and ∞, so that the point of ultimate intersection P coincides with N, some point between F and F'.

3. I shall now consider some misstatements which have been made with reference to this limiting case, and which arise from two causes:

i. *From regarding certain parts of a locus or envelope, described according to a definite law, as the whole.*

ii. *From including in a definite class of curves a number of points which cannot be so included in accordance with the original definition.*

Mr. Ferrers has investigated (*Trilinear Coordinates*, p. 124), the identical relation

$$a^2(p-q)(p-r) + b^2(q-r)(q-p) + c^2(r-p)(r-q) = 4\Delta^2,$$

where p, q, r, are tangential coordinates and Δ the area of the triangle of reference; and in p. 128 it is stated that the equation $\Delta^2 = 0$ represents the two circular points at infinity, from which it is deduced that the equations of all circles may be reduced to the form

$$\Delta^2 - (lp + mq + nr)^2 = 0,$$

where $lp + mq + nr = 0$ is the equation of the centre. But it has been proved that an equation of the form $\mu\nu = 0$ represents the straight line which joins the points $\mu = 0$, $\nu = 0$. Hence the equation $\Delta^2 = 0$ represents not the two circular points at infinity, but rather at a straight line bounded by those points (which we shall call F, F') and having for tangents FF' and *all other straight lines drawn arbitrarily through F or F'.*

Since $lp + mq + nr$ is proportional to the perpendicular distance (ϖ) of the point $lp + mq + nr = 0$, from the straight line whose coordinates are p, q, r, the equation of the circle may be written briefly

$$V \equiv \varpi^2 - k\Delta^2 = 0,$$

where k is some constant quantity.

Now the equation $V = 0$ is satisfied by the coordinates of the common tangents of $\Delta^2 = 0$ and $\varpi^2 = 0$, and therefore represents the fact that every circle satisfies the analytical condition

of touching two straight lines drawn from its centre to the fixed points F, F', a result which might have been anticipated, as the reciprocal of the property expressed by Cartesian coordinates, that every circle may be regarded as passing through two fixed imaginary points at infinity.

The same equation must of course express the fact that all circles pass through the circular points at infinity; this property, however, cannot be deduced quite so directly as the former. The following is the method of proceeding: Let the equation $V=0$ be combined with that of F, so that Δ^2 vanishes; the equation then reduces to $\varpi^2=0$. Hence the two tangents from F to the circle are *coincident*. Consequently F, and likewise F', lie on the circle.

4. Dr. Salmon also (p. 337)* has interpreted a tangential equation of the second degree as representing two conjugate points, and on p. 330, he remarks, in the discussion of a certain problem, "Since $S+kS'$ and the corresponding tangential equation belong to a system of conics passing through four fixed points, the envelope of the system is *nothing but these four points*:" a false conclusion which may be accounted for as follows:

In dealing with Cartesian equations we regard a point as defined by its distances from two fixed straight lines, and accordingly when those distances are both given, the point is accurately determined; it is from the transference of notions with which we have been thus familiarized to cases in which they are inapplicable, that the misstatements to which I have alluded may have arisen. Consider the case of the normal to an ellipse at a point very near to one extremity of the axis: when the normal coincides with the axis, its *ultimate point of intersection* is determinate, being in fact the centre of curvature. Similarly the normal near the other extremity of the axis has a definite point of ultimate intersection distinct from the former, although the lines themselves are coincident throughout their length; and so for a straight line which moves up to coincidence with the axis according to any other given law, there exists a distinct point of ultimate intersection. Hence in discussing tangential equations we are concerned not solely with the actual values of the coordinates

* See also p. 388.
"In general $\Sigma + k\Sigma'$ denotes a conic touching the four tangents common to Σ and Σ'; and when k is determined so that $\Sigma + k\Sigma'$ represents a pair of points, those points are two opposite vertices of the quadrilateral formed by the common tangents."

of any line, which give only the limiting position of that line, but more particularly with the *process* by which the line assumes its final position: for example, the line PT (fig. 13), may move up to coincidence with FF'' in such a manner as always to cut CF produced at a finite distance from C; or it might move up to the same limiting position so as to cut CF'' produced at a finite distance; or again, it may be constantly parallel to FF''. In all of these cases the ultimate position of the line is the same, but its points of ultimate intersection with consecutive lines of the system are different. It is indeed true that every line subject to the equation $\mu\nu = 0$ passes through one of the points $\mu = 0$, $\nu = 0$, since the equation cannot be satisfied unless either $\mu = 0$, or $\nu = 0$, and also that if μ and ν vanish simultaneously the line passes through both of those points; but from the nature of tangential equations the problem is to determine points of *ultimate intersection*, and not merely limiting positions of lines. Where then is the ultimate intersection of two consecutive lines for which μ and ν both vanish? Since there is no reason for concluding that it is F (simply because μ vanishes) which does not point alike to the conclusion that it is F'' because ν vanishes, and since two straight lines cannot have two distinct points of ultimate intersection, it appears that the point to be determined is presented in an indeterminate form, and that it may in accordance with the above investigations be any point on the limited line FF''.

5. The tangential equation $\lambda\mu = 0$ has been shown to represent a straight line terminated at the points $\lambda = 0$, $\mu = 0$, but since a variety of investigations which lead to such equations involve no direct reference to any other portion of the locus than the two points, it may be found convenient to assign an appellation to them. Thus it might be said, that $\lambda = 0$, $\mu = 0$, are the LIMITING POINTS of the equation $\lambda\mu = 0$.

6. M. Chasles in the first part of his *Traité des Sections Coniques*, (p. 31), after the remark that two straight lines may be regarded as a conic in virtue of their being a particular form of the section of a cone by a plane, and also because they can be represented by an equation of the second degree, states that '*le système de deux droites peut être considéré comme formant une section conique*, for no other reason, than that a pair of straight lines possesses certain properties

of conics. The statement is indeed correct; not however for the reason assigned, but because it *happens* to be consistent with other views of the same limiting case. Let us consider the illustration which M. Chasles himself has employed. On a fixed straight line A take four fixed points and join them to a variable point on a second straight line B; the pencil thus formed having a constant anharmonic ratio, we may, it is said, regard A and B as forming a conic section. But from this consideration *alone* we might regard any third straight line as part of the same conic, since the pencil whose vertex is anywhere on the third line, and whose rays pass through the four fixed points, has the same constant anharmonic ratio as the former.

Again, "De même que deux droites peuvent représenter une conique......de même *le système de deux points peut être considéré comme représentant une conique*, quand il ne s'agit que des propriétés relatives aux tangents de ces courbes."

And an illustration corresponding reciprocally to that which was given in the previous instance, is introduced in the following terms:

"Par exemple, quatre droites fixes menée par un de ces deux points rencontrent une droite quelconque menée par l'autre point, en quatre points dont le rapport anharmonique est constant."

But since a straight line drawn through any third point cuts the four lines in a range of the same anharmonic ratio as before, why may not the arbitrary third point in conjunction with the former two be regarded as a conic? Or to go one step further, why may not an infinity of points arranged in an arbitrary manner so as to form any curve whatever, be called in conjunction with the first point 'a conic.'

The following appears to be a satisfactory explanation of the difficulty. If a conic be defined as the envelope of a straight line which cuts four fixed straight lines in a range of constant anharmonic ratio, and the four lines be supposed in their limiting position to meet in a point, the limit of the envelope will be a straight line, the further extremity of which can be determined, if the *way* in which the four lines move up to their limiting position be known, but not otherwise. The straight line joining the two points cuts the pencil at its vertex in a range of indeterminate form, yet of the same value as for any other line, and this joining line may be regarded as a tangent to the limiting conic *at any point* intermediate to the two vertices. Though the property in

question is true for two or any other number of points however situated, yet a collection of conjugate points is not therefore to be called arbitrarily a conic section, whilst if the idea of a *limit* be introduced, the limiting curve will be from this as from other points of view a straight line, and cannot possibly consist of conjugate points.

7. This view of the limiting case may be shown to be in accordance with the ordinary process of reciprocation, for if a hyperbola approach its asymptotes as a limit, then all its tangents coincide with the asymptotes, the points of contact being distributed along the whole extent of the asymptotes. We shall express this briefly, by saying that each asymptote is a tangent to itself at an indeterminate point; but to the tangent at such a point corresponds an *indeterminate* point on the reciprocal which cannot therefore consist simply of two determinate points, but is, in accordance with what precedes, a terminated straight line.

The inclination of the tangent at x', y' to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, being given by the equation $\tan\theta = \frac{b^2}{a^2} \cdot \frac{x'}{y'}$, it follows when the equation of the curve assumes its limiting form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$, that $\tan\theta = \pm\frac{b}{a}$. Hence, the tangents all coincide with the asymptotes, and are not "outre ces deux droites mêmes, toutes celles qui passent par leur point d'intersection."

8. If fixed points P, P' (fig. 14) be joined to any point m on a fixed straight line, and if Pm, $P'm$ cut two fixed straight lines Sa, Sa', in the points a, a', then it may be shown that aa' envelopes a conic section; but if PP' pass through S, then aa' passes through a fixed point H. In this case observes M. Chasles, "Ou peut dire que la conique se reduite au système des deux points S et H." Now, although it is true that all the tangents do in this case pass through S or H, it is not true, as may easily be seen, that all their points of contact coincide with S or H. There is on the contrary an infinity of tangents which ultimately coincide with SH, and have their points of contact distributed throughout the straight line SH. Hence, the straight line SH is the limiting conic.

9. Again, if V (fig. 15) be the vertex of a triangle, whose sides pass through fixed points A, B, and whose base ab has its extremities on two fixed straight lines Oa, Ob, and touches

a conic which has Oa, Ob for tangents, then will the locus of V be a conic. The locus of V is also a conic when the base passes through a fixed point C.* The connexion of the two cases may be thus exhibited. Let the conic which is given in the first case degenerate into the terminated straight line OC, all tangents to which pass through one at least of the points O, C. Of these tangents all that do not pass through C determine one and the same point (O) on the required locus, which point may be obtained independently by means of a tangent which *does* pass through C, viz. in the direction CO. Since then *all* points on the locus may be obtained by merely supposing the base to pass through C, the limiting case may be enunciated in a simplified form. There is, however, no discontinuity in passing from one case to the other, in this or similar investigations, and it will be found that in no case can two points be regarded as a *limit* of a conic. Nor is anything to be gained from asserting arbitrarily, as a kind of supplementary definition, that two points may be called a conic, whilst on the other hand, such departure from analogy tends to obscure the connexion between the general and the limiting forms of problems, which analogy so frequently suggests the method of proceeding from a particular result to others of greater generality.

10. The following method of representing confocal conics, which occurred to me before I had seen Dr. Salmon's articles on the subject (*Conic Sections*, Ed. 4), is investigated in a manner different from that which Dr. Salmon pursued.

Let
$$\varpi \equiv l\lambda + m\mu + n\nu = 0,$$
$$\varpi' \equiv l'\lambda + m'\mu + n'\nu = 0,$$

be the equations of the foci of a system of conics, and let $\{\lambda, \mu, \nu\}^2$ denote, according to Mr. Whitworth's notation, the expression

$$\lambda^2 + \mu^2 + \nu^2 - 2\mu\nu \cos A - 2\nu\lambda \cos B - 2\lambda\mu \cos C,$$

which is known to have a constant value. Then $\varpi\varpi'$ is constant for any conic having the given foci. Hence the equation

$$(l\lambda + m\mu + n\nu)(l'\lambda + m'\mu + n'\nu) = k\{\lambda, \mu, \nu\}^2 \ldots. \text{ (i)},$$

may, if a suitable value be assigned to k, represent any conic

* From this and similar considerations, it might be deduced that *the point C may be regarded as a conic touching Oa, Ob,* as reasonably as it has been inferred elsewhere, that two points may be regarded as a conic.

having the given foci; the conic being an ellipse when k is of the same sign as $\varpi\varpi'$; a hyperbola when k and $\varpi\varpi'$ have different signs.

The two tangents which can be drawn from either of the points $\varpi=0$, $\varpi'=0$, must therefore pass through the limiting points of the equation $\{\lambda, \mu, \nu\}^2=0$. Hence, either focus of a conic may be regarded as the point of intersection of a pair of imaginary tangents which pass through the circular points at infinity.

If (i) reduce to the form

$$l'\left(\frac{\lambda}{a}-\frac{\mu}{b}\right)(l\lambda+m\mu+n\nu)=k\{\lambda, \mu, \nu\}^2 \;\ldots\ldots\; \text{(ii)},$$

it will represent a parabola, the focus $\dfrac{\lambda}{a}-\dfrac{\mu}{b}=0$ being at infinity.

The equation of the circle

$$(l\lambda+m\mu+n\nu)^2=k\{\lambda, \mu, \nu\}^2 \;\ldots\ldots\ldots\; \text{(iii)},$$

expresses the fact that the radius is constant.

11. If the equation

$$\sqrt{(\lambda)}+\sqrt{(\mu)}+\sqrt{(\nu)}=0 \;\ldots\ldots\ldots\ldots\; \text{(iv)},$$

be cleared of radicals and subtracted from $\{\lambda, \mu, \nu\}^2$, the resulting equation will differ by a constant from

$$\mu\nu\sin^2\frac{A}{2}+\nu\lambda\sin^2\frac{B}{2}+\lambda\mu\sin^2\frac{C}{2}=0 \;\ldots\ldots\; \text{(v)},$$

which proves that (iv) and (v) are confocal, since their equations differ by a constant quantity. Hence Mr. Torry's theorem (*Educational Times*, No. 1814).

If a triangle be inscribed in an ellipse and envelope a confocal ellipse, the points of contact lie on the escribed circles of the triangle.

The equations (i), (ii), (iii) may of course be applied to determine the conditions that the general equation of the second degree may represent an ellipse, a parabola, or a hyperbola. The condition that the equation may break up into two factors, is the condition for its representing a terminated straight line.

St. John's College, Cambridge,
March, 1866.

A DEMONSTRATION OF FOURIER'S THEOREM.

By WILLIAM WALTON, M.A., Trinity College.

IN Professor De Morgan's *Differential and Integral Calculus*, p. 618, occurs the following passage, viz.

"We have then

$$\phi x = \frac{1}{2l}\int_{-l}^{+l} \phi v dv + \frac{1}{\pi}\Sigma\left\{\int_{-l}^{+l} \cos w(x-v).\phi v dv \Delta w\right\},$$

which being true for all values of l is true at the limit when l is infinite. Now $\int \phi v dv$ in the first term may increase without limit with l, and $\int \phi v dv : 2l$ may in such case either increase without limit,* have a finite limit, or diminish without limit. If the latter be the case, which it certainly will be whenever $\int_{-\infty}^{+\infty} \phi v dv$ is finite, then, observing that w increases by continually diminishing gradations from 0 to ∞, we have, by the definition of a definite integral,

$$\pi\phi(x) = \int_0^\infty \left\{\int_{-\infty}^{+\infty} \cos w(x-v)\,\phi v dv\right\} dw$$

$$= \int_0^\infty \int_{-\infty}^{+\infty} \cos w(x-v)\,\phi v dw dv;$$

a result usually called Fourier's theorem."

In regard to this passage he has the following foot note:

* "The reasoning of Poisson neglects this limitation, though obvious enough, and Fourier makes a similar apparent error. Poisson makes $\int \phi v dv : 2l$ always vanish when l is infinite: Fourier has missed this term by writing a series $P_1 \cos x + P_2 \cos 2x + \dots$, which should have been $P + P_1 \cos x + P_2 \cos 2x + \dots$. Both are certainly wrong in expression, though the remarks to which I shall presently come remove the limitation, and show the theorem to be universal."

The following demonstration of Fourier's theorem is concise, and, as far as I see, free from objection. The limitation which Professor De Morgan discards, as not being really essential, does not attach itself to my proof, which, as far as I know, is new.

Let λ and μ be both positive quantities, such that $\lambda + x$ and $\mu - x$ are both positive: put

$$y = \int_0^\infty dq \int_{-\lambda}^{\mu} \cos q(\alpha - x)\, \phi(\alpha)\, d\alpha.$$

Integrating both sides with regard to x from $x = 0$ to $x = x$, we have

$$\int_0^x y\,dx = \int_0^\infty \frac{dq}{q} \int_{-\lambda}^{\mu} \{\sin q(x - \alpha) + \sin q\alpha\}\, \phi(\alpha)\, d\alpha$$

$$= \int_0^\infty \frac{dq}{q} \left\{\int_{-\lambda}^{x} + \int_x^{\mu}\right\} \sin q(x - \alpha)\, \phi(\alpha)\, d\alpha$$

$$+ \int_0^\infty \frac{dq}{q} \left\{\int_{-\lambda}^{0} + \int_0^{\mu}\right\} \sin(q\alpha)\, \phi(\alpha)\, d\alpha.$$

But, r being a constant, we have, by a known theorem,

$$\int_0^\infty \frac{dq}{q} \sin(rq) = \pm \frac{\pi}{2},$$

the $+$ or $-$ sign being taken according as r is positive or negative: hence, if x be positive,

$$\int_0^x y\,dx = \frac{\pi}{2} \left\{\int_{-\lambda}^{x} - \int_x^{\mu}\right\} \phi(\alpha)\, d\alpha$$

$$+ \frac{\pi}{2} \left\{\int_0^{\mu} - \int_{-\lambda}^{0}\right\} \phi(\alpha)\, d\alpha$$

$$= \frac{\pi}{2} \left\{\int_{-\lambda}^{0} + \int_0^{x} - \int_x^{\mu}\right\} \phi(\alpha)\, d\alpha$$

$$+ \frac{\pi}{2} \left\{\int_0^{x} + \int_x^{\mu} - \int_{-\lambda}^{0}\right\} \phi(\alpha)\, d\alpha$$

$$= \pi \int_0^x \phi(\alpha)\, d\alpha$$

$$= \pi \int_0^x \phi(x)\, dx.$$

If x be negative,

$$\int_0^x y\,dx = \frac{\pi}{2}\left\{\int_{-\lambda}^{\infty} - \int_x^{\mu}\right\}\phi(\alpha)\,d\alpha$$

$$+\frac{\pi}{2}\left\{\int_0^{\mu} - \int_{-\lambda}^{0}\right\}\phi(\alpha)\,d\alpha$$

$$= \frac{\pi}{2}\left\{\int_{-\lambda}^{\infty} - \int_x^{0} - \int_0^{\mu}\right\}\phi(\alpha)\,d\alpha$$

$$+\frac{\pi}{2}\left\{\int_0^{\mu} - \int_{-\lambda}^{x} - \int_x^{0}\right\}\phi(\alpha)\,d\alpha$$

$$= -\pi\int_x^0 \phi(\alpha)\,d\alpha$$

$$= \pi\int_0^x \phi(x)\,dx.$$

Hence we see that, whether x be positive or negative, $y = \pi\phi(x)$.

The previous reasoning holds good, step by step, if λ and μ be infinitely great. Hence

$$\int_0^{\infty} dq \int_{-\infty}^{+\infty} \cos q(\alpha - x)\,\phi(\alpha)\,d\alpha = \pi\phi(x).$$

The only condition implied in the above proof is that $\phi(\alpha)$ shall not be infinite between the limits: this consideration involves the condition that, when $l = \infty$, the expression

$$\frac{1}{l}\int_{-l}^{+l} \phi(\alpha)\,d\alpha$$

shall not be infinite.

August 30, 1865.

ON INTERPOLATION WITH REFERENCE TO DEVELOPMENT AND DIFFERENTIATION.

(Continued from p. 65.)

By Samuel Roberts, M.A.

13. The example just quoted prepares us for the fact, that in the application of general differentiation to the evaluation of definite integrals, our formulæ cannot be directly used without taking into account the complementary function. Since the theories of M. Liouville and Kelland depend on definite integration, I have postponed till now the comparison of results. It is desirable to exhibit briefly the mode in which the rival formula is deduced.

Since $$\int_0^\infty e^{-ax}\,da = \frac{1}{x},$$

$$\frac{d^\mu}{dx^\mu}\cdot\frac{1}{x} = \int_0^\infty \left(\frac{d}{dx}\right)^\mu \varepsilon^{-ax}da = \int_0^\infty (-a)^\mu \varepsilon^{-ax}da = (-)^\mu\frac{\Gamma(1+\mu)}{x^{1+\mu}}.$$

And generally, since $\int_0^\infty \varepsilon^{-ax}.a^{n-1}da = \Gamma n.x^{-n}$, we have

$$\frac{d^\mu}{dx^\mu}\cdot\frac{1}{x^n} = \frac{(-)^\mu}{\Gamma n}\int_0^\infty \varepsilon^{-ax}.a^{n+\mu-1}da = (-)^\mu\frac{\Gamma n+\mu}{\Gamma n}\,\frac{1}{x^{n+\mu}}\ldots(m),$$

which is the ground-formula.

(1) Now with reference to this process, I have to remark that, as is well known, integration is not always permissible under the sign of definite integration, and consequently it ought not to be assumed, that the process of general differentiation is permissible, until the effect of the process on both members of the equivalence is known, and this is precisely what we have to learn. (2) We can only infer, that the result on the right hand can be reconciled with the result on the left hand by aid of the complementary function.

Take the case of $\int_0^1 x^{m-1}dx = \frac{1}{m}$. Integrate on both sides, then

$$\int_0^1 \frac{x^{m-1}dx}{\log x} = \log m,$$

which is wrong. Of course, we can make it right by means of the complementary function, involving in this case x but not m.

Still, it may be said that, since in the case of integer indices the formula deduced by operating on the integral is correct, we may exercise a discretion in our generalization and determine that what the integral becomes for fractional values, shall be considered the legitimate extension of differentiation. This is so, but then the next step of the actual process is, to determine that the value of the integral shall not govern the value of the differential coefficient for negative fractional indices.

Although, however, this preliminary matter of objection should be removed, the fact that the formula (m) is not correct, or at least is highly inconvenient, in the case of n and $n+\mu$, both integers without the aid of an additional assumption, remains an obstacle.

Transforming (m), we have

$$D^{\mu}x^{m} = (-)^{\mu} \frac{\sin m\pi}{\sin(m-\mu)\pi} \frac{\Gamma 1+m}{\Gamma 1+m-\mu},$$

which Prof. Kelland proposes to correct by writing

$$\sin m\pi \cos\mu\pi \text{ for } \sin(m-\mu)\pi.$$

The inconvenience of the foregoing result can hardly be disputed, and is, it seems to me, a great objection to the process. Yet I do not think it is wrong absolutely, having regard to continuity of form. For let θ be a very small angle, then

$$\sin(2k\pi+\theta)=\theta, \quad \sin\{(2k+1)\pi+\theta\}=-\theta;$$

therefore $$\frac{\sin(m\pi+\theta)}{\sin\{(m-\mu)\pi+\theta\}}=\pm 1,$$

according as μ is even or odd.

Although I do not adopt the result (m) as the fundamental formula of differentiation, it is nevertheless highly important in another sense. For it enables us to generalize definite integrals with great ease and success. So far as the application is concerned, it does not matter whether the left hand result is or is not obtained by means of a complementary function, or whether the operation indicated is differentiation or not. The important fact is that when we

differentiate ϵ^{-ax} under a particular symbol of integration, we must operate in a given manner on the value of the original integral, to obtain a corresponding result.

According to our processes, we have

$$\int_0^\infty \left(\frac{d}{dx}\right)^\mu \epsilon^{-ax} dx = \left(\frac{d}{dx}\right)^\mu \frac{1}{x} + K_1 x^{-\mu-1} + K_2 x^{-\mu-2} + \&c.$$

$$= \left(\frac{\Gamma 0}{\Gamma - \mu} + K\right) x^{-\mu-1} + C.F.$$

Identifying the known result of $(-)^\mu \int_0^\infty \epsilon^{-ax} a^\mu da$ with the foregoing one,

$$K = (-)^\mu \frac{\Gamma 1 + \mu . \Gamma - \mu - \Gamma 0}{\Gamma - \mu}$$

= infinite constant except in particular cases. The value of the definite integral, therefore, bears a relation to the result of (l) similar to that which $\log x$ bears to $\frac{x^0}{0}$ in the integration of $\frac{1}{x}$.

14. It will be observed, that the immediate application of the form (m) to numerical cases, is, after all, no proof of these, independently of the proof of the value of the general integral; for (m) is not proved by differentiation but by the properties of the integral.

In differentiation, the gamma function is finite for negative fractional values of the variable. But in integration, these values must be generally taken as infinite. Originally M. Liouville regarded the gamma function as limited to positive values, even in differentiation, but he subsequently extended its range.

I give one or two examples.

Ex. 1. Since $\int_0^1 x^{m-1} dx = \frac{1}{m}$, we have, by ($m$),

$$\int_0^1 (-\log x)^{p-1} x^{m-1} dx = \frac{\Gamma p}{m^n}.$$

Ex. 2. Since $\int_0^1 \frac{dx}{\{ax + (1-x)\,l\}^2} = \frac{1}{a-l}\left\{\frac{1}{l} - \frac{1}{a}\right\} = \frac{1}{al},$

we have, operating by (m) with regard to a and b successively,

$$\int_0^1 \frac{x^{\alpha-1}(1-x)^{\beta-1}\,dx}{\{ax+(1-x)\,b\}^{\alpha+\beta}} = \frac{\Gamma\alpha\,\Gamma\beta}{\Gamma\alpha+\beta}\cdot\frac{1}{a^\alpha b^\beta}.$$

If $a=b$, this gives Euler's first integral

$$\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\,dx = \frac{\Gamma\alpha\,\Gamma\beta}{\Gamma\alpha+\beta}.$$

If $a-b=1$, we have Abel's form

$$\int_0^1 \frac{x^{\alpha-1}(1-x)^{\beta-1}\,dx}{(x+b)^{\alpha+\beta}} = \frac{\Gamma\alpha\,\Gamma\beta}{\Gamma\alpha+\beta}\cdot\frac{1}{b^\beta(1+b)^\alpha}.$$

Ex. 3. Operating with regard to a to the index $\frac{1}{2}$ on

$$\int_0^\infty \frac{\cos ax}{1+x^2}\,dx = \tfrac{1}{2}\pi\varepsilon^{-a},$$

and observing that, since the result is imaginary on the left hand, we must take an imaginary value on the right, we have

$$\int_0^\infty \frac{x^{\frac{1}{2}}\cos\left(\frac{1}{4}\pi - ax\right)dx}{1+x^2} = \tfrac{1}{2}\pi\varepsilon^{-a}.$$

In treating in a similar manner

$$\int_0^\infty \frac{\cos - ax}{1+x^2}\,dx = \tfrac{1}{2}\pi\varepsilon^{a},$$

a real value results on both sides. (Liouville, *Journal de l'Ecole Polytechnique*, 1832).

In like manner M. Liouville obtains

$$\int_0^\infty \frac{x^\mu \cos\left(ax - \frac{1}{2}\mu\pi\right)dx}{1+x^2} = \tfrac{1}{2}\pi\varepsilon^{-a}.$$

Ex. 4. $\displaystyle\int_0^\infty dx \int_0^\infty dy \int_0^\infty dz\,\varepsilon^{-m(ax+by+cz)} = \frac{1}{abcm^3};$

therefore $\displaystyle\int_0^\infty x^{\alpha-1}dx \int_0^\infty y^{\beta-1}dy \int_0^\infty z^{\gamma-1}dz\,\varepsilon^{-m(ax+by+cz)}\,m^{\alpha+\beta+\gamma-3}dx\,dy\,dz$

$$= \frac{\Gamma\alpha\,\Gamma\beta\,\Gamma\gamma}{a^\alpha b^\beta c^\gamma m^3};$$

therefore $\displaystyle\int_0^\infty x^{\alpha-1} \int_0^\infty y^{\beta-1} \int_0^\infty z^{\gamma-1}\,\varepsilon^{-m(ax+by+cz)}$

$$= \frac{\Gamma\alpha\,\Gamma\beta\,\Gamma\gamma}{a^\alpha b^\beta c^\gamma\,\Gamma\alpha+\beta+\gamma} \times \int_0^\infty \varepsilon^{-mv} v^{\alpha+\beta+\gamma-1}dv,$$

and generally, if ϕv can be expanded according to ascending powers of ε^{-v},

$$\int_0^\infty x^{\alpha-1}\int_0^\infty y^{\beta-1}\int_0^\infty z^{\gamma-1}\ldots\phi\,(ax+by+cz+\ldots)\,dx\,dy\,dz\ldots$$

$$=\frac{1}{a^\alpha b^\beta c^\gamma}\,\frac{\Gamma\alpha\,\Gamma\beta\,\Gamma\gamma\ldots}{\Gamma\alpha+\beta+\gamma+\ldots}\int_0^\infty\phi\,(v)\,u^{\alpha+\beta+\gamma+\ldots-1}.$$

15. The application of general differentiation to the solution of differential equations will be very much the same whichever of the two ground-formulæ be received. It is plain, that the methods will not require change in the case of exponential functions. When the operations affect simple powers, some alteration in form and interpretation will be necessary, according to the method we adopt.

Professor Kelland has so fully entered into this part of the subject in the memoirs cited (*Transactions of the Royal Society of Edinburgh*, vols. XVI. and XX.), that some few examples only are necessary here, in order to shew what kind of modification the present theory makes necessary and to facilitate comparison.

Linear differential equations of fractional orders conform to the close analogy which exists between such equations of integral and positive orders and algebraical equations of corresponding degree.

Thus, if

$$\{(D_x^{\frac{m}{n}})^p+A\,(D_x^{\frac{m}{n}})^{p-1}+\ldots+PD_x^{\frac{m}{n}}+Q\}\,u=0$$

be the equation proposed, the solution may be written as

$$\begin{aligned}u=\;&k_{11}\varepsilon^{\alpha_1\frac{n}{m}x}+k_{12}\varepsilon^{\alpha_2\frac{n}{m}x}+\ldots+k_{1p}\varepsilon^{\alpha_p\frac{n}{m}x}\\&+k_{21}\varepsilon^{w\alpha_1\frac{n}{m}x}+k_{22}\varepsilon^{w\alpha_2\frac{n}{m}x}+\ldots+k_{2p}\varepsilon^{w\alpha_p\frac{n}{m}x}\\&+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\&+k_{m1}\varepsilon^{w^{m-1}\alpha_1\frac{n}{m}x}+k_{m2}\varepsilon^{w^{m-1}\alpha_2\frac{n}{m}x}+\ldots+k_{mp}\varepsilon^{w^{m-1}\alpha_p\frac{n}{m}x},\end{aligned}$$

where $\alpha_1, \alpha_2, \ldots \alpha_p$ are the roots of

$$\alpha^p+A\alpha^{p-1}+\ldots+P\alpha+Q=0,$$

and $1, w, w^m, \ldots w^{m-1}$ are the m^{th} roots of unity.

This result follows from the property $D_x^{\,q}\varepsilon^{ax}=a^q\varepsilon^{ax}$, and the laws of operation for differential symbols and constants.

Usually, however, the solution of differential equations of a fractional order can only be given in series or symbolically.

16. Required the solution of $\left\{1 - mx\left(\frac{d}{dx}\right)^{\frac{1}{2}}\right\}u = 0.$

Here we have

$$u = \left\{1 + mx\left(\frac{d}{dx}\right)^{\frac{1}{2}}\right\}\left\{1 - m^2x\left(\frac{d}{dx}\right)^{\frac{1}{2}}.x\left(\frac{d}{dx}\right)^{\frac{1}{2}}\right\}^{-1}0.$$

But $$m^2x\left(\frac{d}{dx}\right)^{\frac{1}{2}}x\left(\frac{d}{dx}\right)^{\frac{1}{2}} = m^2x\left\{x\frac{d}{dx} + \frac{1}{2}\right\};$$

therefore $$\left\{1 - m^2x\left(\frac{d}{dx}\right)^{\frac{1}{2}}x\left(\frac{d}{dx}\right)^{\frac{1}{2}}\right\}^{-1}0 = Cx^{-\frac{1}{2}}\epsilon^{-\frac{1}{m^2x}},$$

and $$u = \left\{1 + mx\left(\frac{d}{dx}\right)^{\frac{1}{2}}\right\}Cx^{-\frac{1}{2}}\epsilon^{-\frac{1}{m^2x}}.$$

This result will be verified by performing the operation $\left\{1+mx\left(\frac{d}{dx}\right)^{\frac{1}{2}}\right\}$, and afterwards operating with $\left\{1-mx\left(\frac{d}{dx}\right)^{\frac{1}{2}}\right\}$. Thus, first, we have

$$Cx^{-\frac{1}{2}}\epsilon^{-\frac{1}{m^2x}} + mxC\left(\frac{\Gamma\frac{1}{2}}{\Gamma 0\,\Gamma 1}x^{-1} - \frac{\Gamma -\frac{1}{2}}{\Gamma -1\,\Gamma 2}m^{-2}x^{-2} + \frac{\Gamma -\frac{3}{2}}{\Gamma -2\,\Gamma 3}m^{-4}x^{-3} - \ldots\right.$$
$$\left. - \frac{\Gamma\frac{3}{2}}{\Gamma 1\,\Gamma 0}mx^0 + \frac{\Gamma\frac{5}{2}}{\Gamma 2\,\Gamma -1}m^2x - \frac{\Gamma\frac{7}{2}}{\Gamma 3\,\Gamma -2}m^3x \ldots\ldots (m).\right.$$

Secondly, we get

$$Cx^{-\frac{1}{2}}\epsilon^{-\frac{1}{m^2x}} - m^2xC\left\{\frac{\Gamma\frac{1}{2}\,\Gamma 1}{\Gamma 0\,\Gamma 1\,\Gamma\frac{1}{2}}x^{-\frac{1}{2}} - \frac{\Gamma -\frac{1}{2}\,\Gamma 0}{\Gamma -1\,\Gamma 2\,\Gamma -\frac{1}{2}}m^{-2}x^{-\frac{3}{2}}\right.$$
$$+ \frac{\Gamma -\frac{3}{2}\,\Gamma -1}{\Gamma -2\,\Gamma 3\,\Gamma -\frac{3}{2}}m^{-4}x^{-\frac{5}{2}} - \ldots$$
$$\left. - \frac{\Gamma\frac{3}{2}\,\Gamma 2}{\Gamma 1\,\Gamma 0\,\Gamma\frac{3}{2}}mx^{\frac{1}{2}} + \frac{\Gamma\frac{5}{2}\,\Gamma 3}{\Gamma 2\,\Gamma -1\,\Gamma\frac{5}{2}}m^2x^{\frac{3}{2}} - \ldots\right\} = 0.$$

The series in (m) is apparently cypher, at all events, such coefficient vanishes; but it will be seen that the *form* is important.

If we expand the symbolical expression $\left\{1 - mx\left(\frac{d}{dx}\right)^{\frac{1}{2}}\right\}^{-1}0$, we shall get a similar result.

17. The following expressions are useful:

$$\frac{1}{r^\mu}\left\{\frac{1}{x^{r-1}}\frac{d}{dx}\right\}^\mu = \left(\frac{d}{dx^r}\right)^\mu x^{r\left(\frac{m}{r}\right)} = \frac{\Gamma 1+\frac{m}{r}}{\Gamma 1+\frac{m}{r}-\mu}\, x^{m-\mu}.$$

From which also

$$x^{\mu r}\left\{\frac{1}{x^{r-1}}\frac{d}{dx}\right\}^\mu = r^\mu \frac{\Gamma\frac{m}{r}+1}{\Gamma\frac{m}{r}-\mu+1}\, x^m,$$

whence $$x^{\mu r}\left\{\frac{1}{x^{r-1}}\frac{d}{dx}\right\}^\mu \phi(x) = r^\mu \frac{\Gamma\frac{D}{r}+1}{\Gamma\frac{D}{r}-\mu+1}\,\phi(\varepsilon^\theta)\ \ldots.(n),$$

D denoting $\frac{d}{d\theta}$.

This singular equivalence (given by Prof. Kelland in a form adapted to his theory) is an immediate consequence of

$$FD.\varepsilon^{mx} = Fm.\varepsilon^{mx}.$$

If we change the sign of r, we get

$$x^{-\mu r}\left\{x^{r+1}\frac{d}{dx}\right\}^\mu \phi(x) = (-r)^\mu \frac{\Gamma-\frac{D}{r}+1}{\Gamma-\frac{D}{r}-\mu+1}\,\phi(\varepsilon^\theta).$$

To solve $\frac{d^2u}{dx^2} - \frac{i.i+1}{x^2} \pm q^2u = 0$, put ε^θ for x; by which substitution the equation becomes

$$u \pm \frac{q^2}{D+i.D-i-1}\,\varepsilon^{2\theta}u = 0,$$

and we have, assuming $u = \varepsilon^{-i\theta} f(\frac{1}{2}D)\, v$,

$$f\tfrac{1}{2}D\,\{D.D-1.v \pm q^2\varepsilon^{2\theta}v\} = 0,$$

if $$f\tfrac{1}{2}D = \frac{\frac{1}{2}D-\frac{1}{2}}{\frac{1}{2}D-i-\frac{1}{2}}\, f(\tfrac{1}{2}D-1),$$

which is satisfied by

$$f\frac{D}{2} = \frac{\Gamma\frac{1}{2}D+\frac{1}{2}}{\Gamma\frac{1}{2}D-i+\frac{1}{2}} = \varepsilon^\theta\,\frac{\Gamma\frac{1}{2}D+\frac{1}{2}}{\Gamma\frac{1}{2}D-i+1}\,\varepsilon^{-\theta};$$

therefore $u = \varepsilon^{-i\theta} . \varepsilon^{\theta} . \dfrac{\Gamma \frac{1}{2} D + 1}{\Gamma \frac{1}{2} D - i + 1} \varepsilon^{-\theta} \{D . D - 1 \pm q^2 \varepsilon^{2\theta}\} 0$

$$= 2^{-i} . x^{i+1} \left(\frac{1}{x} \frac{d}{dx}\right)^i x^{-1} \{D_x^{\,2} \pm q^{2-1}\} 0.$$

By putting $u = \varepsilon^{-i\theta} F\left(-\dfrac{D}{2}\right)$, we get the form

$$u = x^{-i-1} \left(x^3 \frac{d}{dx}\right)^i x^{-2i+1} (D_x^{\,2} \pm q^2)^{-1} 0.$$

(Boole's *Differential Equations;* Kelland, *Transactions, Royal Society, Edinburgh*).

We may, however, solve the equation in question without directly using the form (n).

For putting z^2 for x, the equation becomes

$$4z^2 \frac{d^2 u}{dz^2} + 2z \frac{du}{dz} = i . i + 1 . u \pm q^2 z u = 0.$$

Then, put $\left(\dfrac{d}{dz} + \dfrac{i+1}{2z}\right)$ for $\dfrac{d}{dz}$, and after division by z, the result is

$$4z \frac{d^2 u'}{dz^2} + (4i + 6) \frac{du'}{dz} . \pm q^2 u' = 0.$$

Again, put $z - \dfrac{i+1}{D_z}$ for z, and the result is

$$4z \frac{d^2 u''}{dz^2} + 2 \frac{du''}{dz} \pm q^2 u'' = 0.$$

Now this is the transformation of $\dfrac{d^2 u}{dx^2} \pm q^2 u = 0$, and since

$$u'' = D_z^{\,-(i+1)} . z^{-\frac{1}{2}i+1} u,$$

we have $\qquad u = x^{i+1} \left(\dfrac{1}{x} \dfrac{d}{dx}\right)^{i+1} \{D_x^{\,2} \pm q^{2-1}\} 0.$

Laplace's equation

$$(1 - \mu^2)^2 \frac{d^2 u}{d\mu^2} - 2\mu (1 - \mu^2) \frac{du}{d\mu} + (n . n + 1 - a^2) u = 0 \;...\; (0),$$

becomes by substitution of $(1 - \mu^2)^{-\frac{1}{2}a} v$ for u, and ε^{θ} for μ,

$$v - \frac{D - a + n - 1 . D - a - n - 2}{D . D - 1} \varepsilon^{2\theta} . v = 0.$$

Let $$v = \frac{1\tfrac{1}{2}D - \tfrac{1}{2}a + \tfrac{1}{2}n + \tfrac{1}{2}}{1\tfrac{1}{2}D - \tfrac{1}{2}a - \tfrac{1}{2}n + \tfrac{1}{2}}\, w,$$

then $$w - \frac{D - a - n - 1 . D - a - n - 2}{D . D - 1}\, \varepsilon^{2\theta} . w = 0.$$

We have, therefore,

$$u = (1-\mu^2)^{-\frac{1}{2}a} . \frac{\Gamma\tfrac{1}{2}D - \tfrac{1}{2}a + \tfrac{1}{2}n + \tfrac{1}{2}}{\Gamma\tfrac{1}{2}D - \tfrac{1}{2}a - \tfrac{1}{2}n + \tfrac{1}{2}}\, w$$

$$= (1-\mu^2)^{-\frac{1}{2}a} . \varepsilon^{a-n+1} . \frac{\Gamma\tfrac{1}{2}D + 1}{\Gamma\tfrac{1}{2}D - n + 1}\, \varepsilon^{-a+n-1} . w$$

$$= (1-\mu^2)^{-\frac{1}{2}a} . x^{2n} . x^{a-n+1} . \left(\frac{1}{x}\frac{d}{dx}\right)^n x^{-a+n-1} . w$$

$$= (1-\mu^2)^{-\frac{1}{2}a} . x^{a+n} . \left(\frac{d}{dx}\frac{1}{x}\right)^n x^{-a+n} \{(1+\mu)^{a+n} k_1 + (1-\mu)^{a+n} k_2\}.$$

Then also we may avoid using the form (n). Setting out from

$$(1+t^2)\frac{d^2u}{dt^2} + t\frac{du}{dt} - a^2 u = 0,$$

put $t + \dfrac{q}{D_t}$ for t, and there results

$$(1+t^2)\frac{d^2u'}{dt^2} + (1+2q)\, t\frac{du'}{dt} + (q^2 - a^2)\, u' = 0.$$

Again, put $D_t + \dfrac{st}{1+t^2}$ for D_t, and there results

$$(1+t^2)\frac{d^2u''}{dt^2} + (1+2q+2s)\, t\frac{du''}{dt}$$
$$+ \left\{q^2 + s - a^2 + (1+2q)\frac{st^2}{1+t^2} - \frac{2st^2}{1+t^2} + \frac{s^2t^2}{1+t^2}\right\} u'' = 0,$$

and if $q = s = nx$,

$$(1+t^2)\frac{d^2u''}{dt^2} + t\frac{du''}{dt} + \left\{\frac{n : n+1}{1+t^2} - a^2\right\} u'' = 0,$$

which by $\dfrac{\mu}{(1-\mu^2)^{\frac{1}{2}}} = t$, becomes the form (0). The solution is readily reduced to Prof. Donkin's form (Boole's *Differential Equations*).

18. This paper has already been so extended, that I must omit any detailed application of the theory to Finite Differences, or to various problems to which such application has been made by Messrs. Kelland and Liouville. These, indeed, are chiefly concerned with the effect of differentiation on the index of the argument x^m, concerning which there is no difference.

M. Cauchy in a memoir "sur un nouveau geure d'integrals," has investigated a definite integral which constitutes an extension of Legendre's applicable to negative values; and by means of it he proves the formula of transformation $\Gamma 1+x\,\Gamma 1-x=\frac{x\pi}{\sin x\pi}$. Before I conclude, it may be as well to notice two objections to Dr. Peacock's form, suggested to Prof. Kelland by Mr. Center. As quoted by the Professor with approval, Mr. Center shews that "without continual introduction of an arbitrary constant, the latter formula (*i.e.* Dr. Peacock's) is inapplicable in many of the most simple cases, *e.g.* in $\frac{d^\mu}{dx^\mu}$ of $\frac{1}{1+x}$ expressed positively, it gives when applied ∞ on one side and not on the other, and when expressed negatively ∞ on both sides; and again it gives for $\frac{d\mu}{dx^\mu}a$ or $\frac{d^\mu}{dx^\mu}ax^0$ the value $\frac{1}{\Gamma 1-\mu}ax^{-\mu}$ which is a function of x when μ is a positive proper fraction." Note, Part III.

Now, with regard to the cases specified, if it may be observed that

$$\frac{1}{1+x}=1-x+x^2, \text{ \&c.},$$

be integrated without reference to the arbitrary constant, we get ∞ on one side and not on the other, and that when we do the same with

$$\frac{1}{1+x}=\frac{1}{x}-\frac{1}{x^2}+\text{ \&c.},$$

we get ∞ on both sides; and as to the second objection, why is it less credible, that $D_x^\mu ax^0$ should be a function of x, than that $D^{-n}.0$ should be so?

It has not been my object to set forth the foregoing method of procedure as the only consistent one. Both ground-formulæ are included in the general functional solution, while the general solution is so vast in its compass, it would probably be of little use, if it could be given. Probably it would

be found a sufficiently laborious task to give the full extent and meaning of

$$\frac{d^n}{dx^n}(1+x)^m(a_0+a_1x+a_2x^2+\ldots)^p.$$

I leave this and other more general forms where the indices proceed by the difference $\pm q$ instead of ± 1, with the feeling, that, whatever theoretical interest may attach to general differentiation, it cannot yet be said to be of much practical utility.

June, 1865.

ON A THEOREM IN QUADRICS.

By Frederick Purser, M.A., Trinity College, Dublin.

The condition that two quadrics should be such that a tetrahedron can be inscribed in one having two pairs of opposite edges on the surface of the other, has been given by Dr. Salmon in his *Geometry of Three Dimensions*, Art. 198. This, with its reciprocal, he considers analogous to the condition that two plane conics should be such that a triangle may be inscribed in one and circumscribed to the other. Following this analogy, I propose, in the present paper, to discuss the problem: *Two pairs of opposite edges of a tetrahedron lie on the surface of one quadric*

$$Ax^2+By^2+Cz^2+Dw^2=0 \;\ldots\ldots\ldots\ldots\; (V),$$

and three of its faces touch another quadric

$$ax^2+by^2+cz^2+dw^2=0 \;\ldots\ldots\ldots\ldots\ldots\; (U);$$

it is required to find the envelope of the fourth face.

Let $ABCD$ be the tetrahedron, AB, AC, BD, CD being generatrices of V, and the planes ABD, ACD, BCD tangent planes to U. Now, since ABD, which is evidently a tangent plane to V at B, also touches U, B must lie on the quadric

$$\frac{A^2}{a}x^2+\frac{B^2}{b}y^2+\frac{C^2}{c}z^2+\frac{D^2}{d}w^2=0 \;\ldots\ldots\ldots\; (S),$$

which is the locus of poles with respect to V of tangent planes to U. We may then restate the problem as follows: *Through any point x', y', z', w' in the curve of intersection of V and S there are drawn two generatrices of V. Through each of these a tangent plane is drawn to U distinct from the plane of the two generatrices. These two planes will meet V again in two generatrices of opposite systems, and which therefore lie in one plane (R). The envelope of this plane is required.*

The condition that the line

$$\left.\begin{array}{l} \alpha x + \beta y + \gamma z + \delta w = 0 \\ \alpha' x + \beta' y + \gamma' z + \delta' w = 0 \end{array}\right\},$$

should meet the curve U, V, is (see Salmon's *Geometry of Three Dimensions*, Art. 208) $\sigma^2 - 4\rho\rho' = 0$, where

$$\sigma = \left(\frac{1}{aB} + \frac{1}{Ba}\right)(\gamma\delta' - \gamma'\delta)^2 + \&c.,$$

$$\rho = \frac{1}{ab}(\gamma\delta' - \gamma'\delta)^2 + \&c.,$$

$$\rho' = \frac{1}{AB}(\gamma\delta' - \gamma'\delta)^2 + \&c.$$

Hence the equation of the four common tangent planes to U, V from any point x', y', z', w' is

$$4ABCDabcd(UU' - P^2)(VV' - Q^2) - \Phi^2 = 0 \dots \text{(I)},$$

where
$$P = axx' + byy' + czz' + dww',$$
$$Q = Axx' + Byy' + Czz' + Dww',$$
$$\Phi = CcDd(Cd + cD)(xy' - x'y)^2 + \&c.$$

Now when $V' = 0$ and $S' = 0$, it is not difficult to see that Φ breaks up into two factors, one of which is Q, and the other $Axx'(aH - Abcd) + \&c.$, where H is an abbreviation for $bcD + bdC + cdB$. Equation (I) will then be divisible by Q^2, leaving as the equation of the two planes drawn as in the second statement of the problem

$$4ABCDabcd(UU' - P^2) + \{Axx'(aH - Abcd + \&c.\}^2 = 0 \dots\dots \text{(II)}.$$

Let the separate equations of these planes be

$$\lambda x + \mu y + \nu z + \rho w = 0,$$

and
$$\lambda' x + \mu' y + \nu' z + \rho' w = 0.$$

Then the equation of the pair of planes passing through their intersection with V, is readily seen to be

$$\left(\frac{\lambda\lambda'}{A}+\frac{\mu\mu'}{B}+\frac{\nu\nu'}{C}+\frac{\rho\rho'}{D}\right)V$$
$$-2(\lambda x+\mu y+\nu z+\rho w)(\lambda' x+\mu' y+\nu' z+\rho' w)=0 \;\ldots\ldots\; \text{(III)}.$$

In this equation the coefficients of x^2, y^2, z^2, w^2 are proportional to

$$A\left(\frac{\mu\mu'}{B}+\frac{\nu\nu'}{C}+\frac{\rho\rho'}{D}-\frac{\lambda\lambda'}{A}\right),\quad B\left(\frac{\nu\nu'}{C}+\frac{\rho\rho'}{D}+\frac{\lambda\lambda'}{A}-\frac{\mu\mu'}{B}\right),$$
$$C\left(\frac{\rho\rho'}{D}+\frac{\lambda\lambda'}{A}+\frac{\mu\mu'}{B}-\frac{\nu\nu'}{C}\right),\quad D\left(\frac{\lambda\lambda'}{A}+\frac{\mu\mu'}{B}+\frac{\nu\nu'}{C}-\frac{\rho\rho'}{D}\right).$$

If in these we substitute for $\lambda\lambda'$, $\mu\mu'$, $\nu\nu'$, $\rho\rho'$ their values derived from (II), and reduce by means of the conditions

$$V'=Ax'^2+By'^2+Cz'^2+Dw'^2=0,$$
$$S'=\frac{A^2}{a}x'^2+\frac{B^2}{b}y'^2+\frac{C^2}{c}z'^2+\frac{D^2}{d}w'^2=0,$$

they become proportional to

$$A^2x'^2\{P+4\alpha(\beta+\gamma+\delta-\alpha)\},\quad B^2y'^2\{P+4\beta(\gamma+\delta+\alpha-\beta)\},$$
$$C^2z'^2\{P+4\gamma(\delta+\alpha+\beta-\gamma)\},\quad D^2w'^2\{P+4\delta(\alpha+\beta+\gamma-\delta)\},$$

where α, β, γ, δ denote $\frac{A}{a}$, $\frac{B}{b}$, $\frac{C}{c}$, $\frac{D}{d}$, and

$$P=\alpha^2+\beta^2+\gamma^2+\delta^2-2\alpha\beta-2\alpha\gamma-2\alpha\delta-2\beta\gamma-2\beta\delta-2\gamma\delta.$$

Now equation (III) is manifestly equivalent to $QR=0$, R being the plane whose envelope is sought. Hence, the equation of R will be

$$Axx'\{P+4\alpha(\beta+\gamma+\delta-\alpha)\}+Byy'\{P+4\beta(\gamma+\delta+\alpha-\beta)\}$$
$$+Czz'\{P+4\gamma(\delta+\alpha+\beta-\gamma)\}+Dww'\{P+4\delta(\alpha+\beta+\gamma-\delta)\}=0.$$

Now $\qquad \{P+4\alpha(\beta+\gamma+\delta-\alpha)\}^2=M+8\alpha N,$

where $M=P^2-64\alpha\beta\gamma\delta$, $N=(\alpha+\beta+\gamma+\delta)P+8\Sigma\alpha\beta\gamma$.

Similarly $\qquad \{P+4\beta(\gamma+\delta+\alpha-\beta)\}^2=M+8\beta N,$

$$\{P+4\gamma(\delta+\alpha+\beta-\gamma)\}^2=M+8\gamma N,$$

and $\qquad \{P+4\delta(\alpha+\beta+\gamma-\delta)\}^2=M+8\delta N.$

Now the condition that any plane $lx+my+nz+rw=0$, should touch a quadric given by the equation

$$px^2+qy^2+sz^2+tw^2=0,$$

is

$$\frac{l^2}{p}+\frac{m^2}{q}+\frac{n^2}{s}+\frac{r^2}{t}=0.$$

Hence, on account of the condition $S'=0$, the plane R will always touch the quadric $MU+8NV=0$, a quadric of the form $U+\lambda V$. This result appears analogous to the theorem in plane conics: *If from any point on one conic V, a pair of tangents be drawn to another conic U, the envelope of the line joining the points where these meet V again is a conic of the form $U+\lambda V$.* If $N=0$, the quadric $MU+8NV=0$ reduces to $U=0$. This condition then should be identical with the second condition given by Dr. Salmon, *Geometry of Three Dimensions*, Art. 198. That it is so, is immediately evident, if we remember that α, β, γ, δ are the roots in μ of the equation $\Delta(\mu U-V)=0$. Again, since $V'=0$, the plane R will always touch the quadric $MV+8NS=0$. In the particular case in which $M=0$, this reduces to $S=0$. Now the plane R is always a tangent plane to V; in this case then its point of contact with V will lie on U. The conditions $M=0$, $N=0$, may be written in the simpler forms $\alpha^{\frac{1}{2}}+\beta^{\frac{1}{2}}+\gamma^{\frac{1}{2}}+\delta^{\frac{1}{2}}=0$, and

$$(\alpha+\beta-\gamma-\delta)(\alpha+\gamma-\beta-\delta)(\alpha+\delta-\beta-\gamma)=0,$$

respectively. From the preceding results and their reciprocals may be deduced the following theorems: *If at any point P on the curve of intersection of two quadrics U, V two generatrices PQ, PR be drawn, meeting the curve UV again in Q, R, then the other generatrices of V drawn at Q and R will intersect on the curve VH, where H is a fixed quadric inscribed in the same developable with U and V.* If we suppose, for example, U and V to be confocal surfaces, the theorem becomes: *If from any point on a line of curvature of a quadric two generatrices be drawn, the other generatrices through the points where these meet the line of curvature again will intersect on a fixed line of curvature.** If the quadric be given in the form

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1=0,$$

* This last result would also follow immediately from the consideration that the generatrices form a limiting case of geodetics touching the same line of curvature.

this fixed line of curvature will be the same as the original, provided

$$\frac{1}{b^2}+\frac{1}{c^2}-\frac{1}{a^2}=0, \text{ or } \frac{1}{c^2}+\frac{1}{a^2}-\frac{1}{b^2}=0, \text{ or } \frac{1}{a^2}+\frac{1}{b^2}-\frac{1}{c^2}=0.$$

For then the condition reciprocal to $N=0$ will obtain between

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1=0 \text{ ....} (V),$$

and any confocal

$$\frac{x^2}{a^2-h^2}+\frac{y^2}{b^2-h^2}+\frac{z^2}{c^2-h^2}-1=0 \text{ ....} (U).$$

If the condition $M=0$ be fulfilled between any two quadrics V and U, we deduce from preceding results the theorem, *If at any point on the curve VS a pair of generatrices of V be drawn, then two tangent planes to U drawn one through each generatrix will meet on U.* Hence, also, *If at any point on the curve UV, a pair of generatrices of V be drawn, two tangent planes may be taken to U, one through each generatrix which shall meet on VS.*

Another application of the equation of the plane (R), is to find the condition that two quadrics U, V, may be such, that a hexahedron can be described, six of whose edges lie on V, and six of whose faces touch U. For, denoting one system of generatrices of V by $PQRST$, &c., and the other by P', Q', R', S', &c., let there be drawn the successive planes PQ', $Q'R$, RS', $S'T$, TU', $U'P$, all of which are supposed to be tangent planes to the quadric U. Now, if we denote the point PU' by x', y', z', w', and RS' by x'', y'', z'', w'', it follows from what has been proved, that the equation of the plane TQ' may be written in either of the forms

$$Axx'\{P+4\alpha(\beta+\gamma+\delta-\alpha)\}+Byy'\{P+4\beta(\gamma+\delta+\alpha-\beta)\}$$
$$+Czz'\{P+4\gamma(\delta+\alpha+\beta-\gamma)\}+Dww'\{P+4\delta(\alpha+\beta+\gamma-\delta)\}=0,$$
$$Axx''\{P+4\alpha(\beta+\gamma+\delta-\alpha)\}+\text{\&c.}+\text{\&c.}=0.$$

Now these equations cannot be identical, unless

$$P+4\alpha(\beta+\gamma+\delta-\alpha)=0,$$

and $y':z':w'::y'':z'':w''$, or $P+4\beta(\gamma+\delta+\alpha-\beta)=0$,

and $\quad z':x':w'::z'':x'':w''$, or &c.

Now if $P+4\alpha(\beta+\gamma+\delta-\alpha)=0$, we have, as before, by

squaring, $M+8aN=0$. Hence, the required condition is obtained by eliminating a between

$$M+8aN=0, \text{ and } P+4ap-8a^2=0,$$

where $p=\alpha+\beta+\gamma+\delta$. The result is

$$M^2+4pMN-8N^2=0.$$

This can be at once expressed in terms of the invariants of the two quadrics, by writing for M, p, N, their values, viz.

$$M=\frac{(\Theta^2-4\Delta\Phi)^2-64\Delta^3\Delta'}{\Delta^4},$$

$$p=-\frac{\Theta}{\Delta},\ N=-\frac{\Theta.(\Theta^2-4\Delta\Phi)+8\Delta^2\Theta'}{\Delta^3}.$$

It may be worth while to notice the relations which connect the invariant conditions between V and U with those between V and S and between S and U. In fact, if we denote as before the roots of $\Delta(\lambda U-V)$ in λ by $\alpha, \beta, \gamma, \delta$, those of $\Delta(\lambda V-S)$ will also be $\alpha, \beta, \gamma, \delta$, and those of $\Delta(\lambda U-S)$ will be $\alpha^2, \beta^2, \gamma^2, \delta^2$. Hence, if the relation $\Theta=0$ holds for V and S, the relation $\Theta=0$ holds for U and V, and the relation $M=0$ for U and S. If again the relation $N=0$ hold for V and S, the relation $M=0$ will hold for U and S, and the relation $N=0$ for U and V. Lastly, if the relation $M=0$ hold for V and S, it holds for U and V.

Blackrock, Co. Dublin,
July, 1866.

THREE BRIEF NOTES ON QUESTIONS IN ANALYTIC GEOMETRY.

By W. F. Walker, M.A.

Note I. It is proposed to apply the formula (published in the June Number of the *Quarterly Journal*) for the area of the triangle formed by three lines, the equations of which are referred to trilinear coordinates; to express the area of the triangle of the polars of the middle points of the sides of a given triangle with respect to an inscribed conic.

The expression thus obtained, compared with the known result that the area in question is equal to that of the original triangle, affords an indirect proof of a somewhat curious determinant identity.

If the equations of three lines referred to trilinear coordinates are $\{l_1.ax + m_1.by + n_1.cz = 0\}$, etc.; the area of the triangle formed by them is

$$\left[(S).\frac{(l_1 m_2 n_3)^2}{P_1 P_2 P_3}\right];$$

where S is the area of the fundamental triangle and

$$P_1 = \begin{vmatrix} 1, & 1, & 1 \\ l_2, & m_2, & n_2 \\ l_3, & m_3, & n_3 \end{vmatrix};$$

Let P, Q, R, the points of bisection of the sides of a triangle ABC, be the vertices of the fundamental triangle; then evidently any conic inscribed in ABC is of the form

$$l^2(by+cz)^2 + m^2(cz+ax)^2 + n^2(ax+by)^2$$
$$-2mn(cz+ax)(ax+by)\ldots = 0 \ldots\ldots\ldots (1),$$

which, on development, gives (putting, $mn + nl + lm = \theta$)

$$(m-n)^2 a^2x^2 + (n-l)^2 b^2y^2 + (l-m)^2 c^2z^2$$
$$-2(\theta - l^2).bcyz - 2(\theta - m^2).cazx - 2(\theta - n^2).abxy = 0 \ldots (2).$$

The equations of the polars of P, Q, R are therefore

$$(m-n)^2.ax + (n^2-\theta).by + (m^2-\theta).cz = 0,$$
$$(n^2-\theta).ax + (n-l)^2.by + (l^2-\theta).cz = 0,$$
$$(m^2-\theta).ax + (l^2-\theta).by + (l-m)^2.cz = 0,$$

whence using the formula for the area given above, we find

$$(\text{polar area}) = [PQR]\,\frac{\begin{Bmatrix} (m-n)^2, & n^2-\theta, & m^2-\theta \\ n^2-\theta, & (n-l)^2, & l^2-\theta \\ m^2-\theta, & l^2-\theta, & (l-n)^2 \end{Bmatrix}^2}{\begin{Bmatrix} 1, & 1, & 1 \\ n^2-\theta, & (n-l)^2, & l^2-\theta \\ m^2-\theta, & l^2-\theta, & (l-m)^2 \end{Bmatrix}.\begin{Bmatrix} (m-n)^2, & n^2-\theta, & m^2-\theta \\ 1, & 1, & 1 \\ m^2-\theta, & (l^2-\theta), & (l-m)^2 \end{Bmatrix}.\begin{Bmatrix} (m-n)^2, & n^2-\theta, & m^2-\theta \\ n^2-\theta, & (n-l)^2, & l^2-\theta \\ 1, & 1, & 1 \end{Bmatrix}}.$$

Now it has been proved that (polar area) $= ABC = 4\,(FQR)$. Hence is derived the determinant identity

$$\begin{vmatrix} \lambda, & \nu', & \mu' \\ \nu', & \mu, & \lambda' \\ \mu', & \lambda', & \nu \end{vmatrix}^2 = 4 \begin{Bmatrix} 1, & 1, & 1 \\ \nu', & \mu, & \lambda' \\ \mu', & \lambda', & \nu \end{Bmatrix} . \begin{Bmatrix} \lambda, & \nu', & \mu' \\ 1, & 1, & 1 \\ \mu', & \lambda', & \nu \end{Bmatrix} . \begin{Bmatrix} \lambda, & \nu', & \mu' \\ \nu', & \mu, & \lambda' \\ 1, & 1, & 1 \end{Bmatrix},$$

in which, for brevity, we have written

$$(m-n)^2 = \lambda,\ \ (n-l)^2 = \mu,\ \ (l-m)^2 = \nu,\ \ mn + nl + lm = \theta,$$
$$l^2 - \theta = \lambda',\ \ m^2 - \theta = \mu',\ \ n^2 - \theta = \nu',$$

the numbers l, m, n being arbitrary.

NOTE II. The equation of the line at infinity in trilinear coordinates is $ax + by + cz = 0$. This result may be very simply established as follows:

The general equation of a right line being written in the form $l.ax + m.by + n.cz = 0$, it is obvious that the ratios $m:n$, $n:l$, $l:m$ give us the ratios in which the sides of the fundamental triangle are cut by the line. Now the line at infinity cuts the three sides externally, each in a ratio of equality; for it therefore, the numbers l, m, n, are all equal; therefore, &c. It is convenient to remember that in the same general form of equation the ratios $l:m:n$ give the ratios of the perpendiculars from the vertices of the triangle on the line; *i.e.* l, m, n are then proportional to the three-point coordinates of the line.

NOTE III. The general tangential equation of the two circular points at infinity may be thus derived:

The condition of perpendicularity for two lines in trilinear coordinates is

$$\lambda\lambda' + \mu\mu' + \nu\nu' - (\mu\nu' + \mu'\nu)\cos A$$
$$- (\nu\lambda' + \nu'\lambda)\cos B - (\lambda\mu' + \lambda'\mu)\cos C = 0.$$

Now it is an obvious property of any line through either circular point, that the perpendicular to it coincides with the line itself; hence put $\lambda' = \lambda$, &c., we have, for the required equation, at once

$$\lambda^2 + \mu^2 + \nu^2 - 2\mu\nu\cos A - 2\nu\lambda\cos B - 2\lambda\mu\cos C = 0.$$

9, Trinity College, Dublin,
July, 1865.

NOTES ON DETERMINANTS.

By JOSEPH HORNER, M.A. of Clare College.

BALTZER has shewn (II. 6) that any determinant may be put into the form of another of higher degree. In what follows I pass to one of lower degree by a method analogous to the well known process of elimination by equalizing coefficients and subtracting, a method which neither Baltzer nor Brioschi seems to have had occasion for in its completeness. Though too simple and obvious in itself to call for much remark, one or two of its results will probably be interesting.

Ex. I. Let $R = \begin{vmatrix} a_{1,1} \dots a_{1,n} \\ \vdots \\ a_{n,1} \dots a_{n,n} \end{vmatrix}$ and $\pi = a_{1,q}\ a_{2,q} \dots a_{n,q}$, also $b_{r,s} = \dfrac{a_{r,s}}{a_{r,q}}$, and therefore $b_{r,q} = 1$. Then

$$R = \pi \begin{vmatrix} b_{1,1} \dots b_{1,q-1} & 1 & b_{1,q+1} \dots b_{1,n} \\ \vdots & & \\ b_{n,1} \dots b_{n,q-1} & 1 & b_{n,q+1} \dots b_{n,n} \end{vmatrix}.$$

Subtract the p^{th} row from each of the others, and reduce as in Baltzer, II. 5. Then

$$R = (-)^{p+q}\pi \begin{vmatrix} b_{1,1} - b_{p,1} \dots b_{1,q-1} - b_{p,q-1},\ b_{1,q+1} - b_{p,q+1} \dots b_{1,n} - b_{p,n} \\ \vdots \\ b_{p-1,1} - b_{p,1} \dots b_{p-1,q-1} - b_{p,q-1},\ b_{p-1,q+1} - b_{p,q+1} \dots b_{p-1,n} - b_{p,n} \\ b_{p+1,1} - b_{p,1} \dots b_{p+1,q-1} - b_{p,q-1},\ b_{p+1,q+1} - b_{p,q+1} \dots b_{p+1,n} - b_{p,n} \\ \vdots \\ b_{n,1} - b_{p,1} \dots b_{n,q-1} - b_{p,q-1},\ b_{n,q+1} - b_{p,q+1} \dots b_{n,n} - b_{p,n} \end{vmatrix}.$$

Now $b_{r,s} - b_{p,s} = a_{p,q}\ a_{r,q} \begin{vmatrix} a_{p,q}, & a_{p,s} \\ a_{r,q}, & a_{r,s} \end{vmatrix} = a_{p,q}\ a_{r,q}\ (p, q, r, s)$,

as we may write. Whence

$$R = \frac{(-)^{p+q}}{a_{p,q}^{\,n-2}}$$

$$\begin{vmatrix} (p, q, 1, 1) \dots (p, q, 1, q-1)(p, q, 1, q+1) \dots (p, q, 1, n) \\ \vdots \\ (p, q, p-1, 1) \dots (p, q, p-1, q-1)(p, q, p-1, q+1) \dots (p, q, p-1, n) \\ (p, q, p+1, n) \dots (p, q, p+1, q-1)(p, q, p+1, q+1) \dots (p, q, p+1, n) \\ \vdots \\ (p, q, n, 1) \dots (p, q, n, q-1)(p, q, n, q+1) \dots (p, q, n, n) \end{vmatrix}$$

If $a_{r,s} = d^r x_s$, the equation

$$\begin{vmatrix} (1,1,2,2) & (1,1,2,3) \\ (1,1,3,2) & (1,1,3,3) \end{vmatrix} = a_{1,1} \begin{vmatrix} a_{1,1} & a_{1,3} \\ a_{3,1} & a_{3,3} \end{vmatrix}$$

becomes

$$\begin{Vmatrix} \begin{vmatrix} dx_1 & dx_2 \\ d^2x_1 & d^2x_2 \end{vmatrix} & \begin{vmatrix} dx_1 & dx_3 \\ d^2x_1 & d^2x_3 \end{vmatrix} \\ \begin{vmatrix} dx_1 & dx_2 \\ d^3x_1 & d^3x_2 \end{vmatrix} & \begin{vmatrix} dx_1 & dx_3 \\ d^3x_1 & d^3x_3 \end{vmatrix} \end{Vmatrix} = dx_1 \begin{vmatrix} dx_1 & dx_2 & dx_3 \\ d^2x_1 & d^2x_2 & d^2x_3 \\ d^3x_1 & d^3x_2 & d^3x_3 \end{vmatrix},$$

a result used in Gregory's *Solid Geometry*, Art. 266, but which may also be obtained from the adjunct system.

As another application let $R = \begin{vmatrix} a_{1,2} \dots a_{1,4} \\ \vdots \\ a_{4,1} \quad a_{4,4} \end{vmatrix}$ with the conditions $a_{r,s} = a_{s,r}^{-1}$ and $a_{r,r} = \pm 1$. Then

$$(q, p, s, r) = \frac{-(p, q, r, s)}{a_{p,q} a_{p,s} a_{r,q} a_{r,s}},$$

and therefore $(p, p, r, r) = 0$. Hence if $p = 1$ and $q = 1$,

$$R = \begin{vmatrix} 0 & (1,1,2,3) & (1,1,2,4) \\ (1,1,3,2) & 0 & (1,1,3,4) \\ (1,1,4,2) & (1,1,4,3) & 0 \end{vmatrix}$$

$$= \frac{-(a_{1,3} \mp a_{1,2} a_{2,3})(a_{1,4} \mp a_{1,2} a_{2,4})(a_{1,4} \mp a_{1,3} a_{3,4})(a_{2,4} \mp a_{2,3} a_{3,4})}{a_{1,2} a_{1,3} a_{1,4} a_{2,3} a_{2,4} a_{3,4}}.$$

So that in this instance R can be resolved into factors, and the last expression has the property of reproducing itself when its elements are replaced by their reciprocals.

Ex. II. Let R_n be the skew symmetrical determinant of even dimensions. We may prove in a simpler manner than has hitherto been done, that it is the square of a rational function of the elements.

We have $R_n = \begin{vmatrix} a_{1,1} \dots a_{1,n} \\ \vdots \\ a_{n,1} \dots a_{n,n} \end{vmatrix}$, where n is even, $a_{q,p} = -a_{p,q}$, and $a_{p,p} = 0$. Put $\dfrac{a_{p,q}}{a_{p,1} \; a_{1,q}} = b_{p,q}$, and therefore $b_{q,p} = -b_{p,q}$, and $b_{p,p} = 0$.

Dividing the rows in order by $1, a_{2,1}, \dots a_{n,1}$, and the columns by $-1, a_{1,2}, \dots a_{1,n}$, R becomes

$$-(a_{1,2} \dots a_{1,n})(a_{2,1} \dots a_{n,1}) R_{n-2} = (a_{1,2} \dots a_{1,n})^2 R_{n-2} = K_2^2 R_{n-2} \text{ (say)},$$

where
$$R_{n-2}=\begin{vmatrix} 0 & 1 & 1 & \dots 1 \\ -1 & 0 & b_{2,3} & b_{2,n} \\ -1 & b_{3,2} & 0 & b_{3,n} \\ \vdots & & & \\ -1 & b_{n,2} & b_{n,3} & 0 \end{vmatrix}.$$

Let
$$c_{r,s}=b_{r,s+1}-b_{r,s},$$
and
$$d_{r,s}=c_{r+1,s}-c_{r,s}=b_{r+1,s+1}-b_{r+1,s}-b_{r,r+1}+b_{r,s}.$$
Therefore
$$d_{s,r}=b_{s+1,r+1}-b_{s+1,r}-b_{s,r+1}+b_{s,r}=-d_{r,s},$$
and
$$d_{r,r}=-b_{r+1,r}-b_{r,r+1}=0.$$

In R_{n-2} write

for the 3rd row the 3rd – the 2nd, element by element,

.... 4th .... 4th – ... 3rd, ...,

⋮

.... nth .... nth – ... $(n-1)$th, ...,

and then reduce as in Baltzer, II. 5. Thus we have
$$R_{n-2}=\begin{vmatrix} 1 & 1 & \dots 1 \\ c_{2,2} & c_{2,3} & c_{2,n} \\ \vdots & & \\ c_{n-1,2} & c_{n-1,3} & c_{n-1,n} \end{vmatrix}.$$

Here,

for the 2nd column write the 2nd – the 1st, element by element,

.... 3rd .... 3rd – ... 2nd, ...,

⋮

.... nth .... nth – ... $(n-1)$th ....,

and reduce as before. Whence
$$R_{n-2}=\begin{vmatrix} d_{2,2} & \dots d_{2,n-1} \\ \vdots & \\ d_{n-1,2} & \dots d_{n-1,n-1} \end{vmatrix}.$$

And this by the relations above established is a skew symmetrical function of $(n-2)$ dimensions. So that we may write
$$R_n=K_2^2R_{n-2},\quad R_{n-2}=K_4^2R_{n-4},\quad R_{n-4}=K_6^2R_{n-6},\dots R_4=K_{n-2}^2R_2;$$
where $K_4=(d_{2,3}\dots d_{2,n-1})$, and so forth. R_2 is a determinant of the form $\begin{vmatrix} 0 & K_n \\ -K_n & 0 \end{vmatrix}$, and therefore $=K_n^2$. Upon the whole then $R_n=(K_2\ K_4\dots K_n)^2$ where the quantities in the right-hand member are rational functions of the elements,

which are each readily derivable from its immediate predecessor.

Arithmetical Product of Determinants. The usual formula for the product of determinants introduces many terms which cancel each other, but to the presence of which the formula owes its analytical utility. For arithmetical purposes however the product may be given a very convenient arrangement, wherein those terms do not appear.

Let $\quad P = \Sigma \pm a_{1,1} \ldots a_{n,n}, \quad Q = \Sigma \pm b_{1,1} \ldots b_{n,n}.$

While the second suffixes of the elements b undergo all their permutations by interchanging one pair at a time, any one term of Q (as the r^{th}) is transformed into every other term of it successively, the signs being alternate. It follows therefore that multiplying the r^{th} term of P by Q is the same thing as multiplying that term by the r^{th} terms of the several determinants

$$\Sigma \pm b_{1,1} \; b_{2,2} \ldots b_{n,n}, \; \Sigma \pm b_{1,2} \; b_{2,1} \ldots b_{n,n} \ldots, \; \Sigma \pm b_{1,1} \; b_{2,2} \ldots b_{n-1,n},$$

and taking their sum. Consequently, the product PQ may be obtained by forming the products of the 1^{st}, 2^{nd}, ...n^{th} terms of P, with the 1^{st}, 2^{nd}, ...n^{th} terms respectively of the determinants

$$\Sigma \pm b_{1,1} \; b_{2,2} \ldots b_{n,n}, \; \Sigma \pm b_{1,2} \; b_{2,1} \ldots b_{n,n} \ldots \Sigma \pm b_{1,1} \; b_{2,2} \ldots b_{n-1,n},$$

and taking their sum.

Now the products of the 1^{st}, 2^{nd}, ...n^{th} terms of P with the corresponding ones of Q are the positive quantities

$$a_{1,1} \; b_{1,1} . a_{1,2} \; b_{1,2} \ldots a_{n,n} \; b_{n,n}; \; a_{1,2} \; b_{1,2} . a_{2,1}, \; b_{2,1} \ldots a_{n,n} \; b_{n,n} \ldots;$$

which may be arranged like a determinant in the form

$$\begin{array}{|l} \overline{a_{1,1}b_{1,1} \ldots a_{1,n}b_{1,}b_{n}} \\ \vdots \qquad\quad \vdots \\ a_{n,1}b_{n,1} \ldots a_{n,n}b_{n,n}, \end{array}$$

the distinction being that, unlike a determinant, all its terms are positive. Such expressions might, for want of a word, be called conterminants, and distinguished from determinants by the mark $\ulcorner$ instead of $|\;|$, or by using the form

$$\Sigma a_{1,1} \ldots a_{n,n} \text{ instead of } \Sigma \pm a_{1,1} \ldots a_{n,n}.$$

From the first conterminant in PQ we obtain the rest of the product by permuting the second suffixes of the factors b

in pairs one at a time; so that we thus get a series of conterminants alternately + and −. Hence

$$PQ = \overline{\left| \begin{matrix} a_{1,1}b_{1,1} & a_{1,2}b_{1,2} \ldots a_{1,n}b_{1,n} \\ \vdots & \\ a_{n,1}b_{n,1} & a_{n,2}b_{n,2} \ldots a_{n,n}b_{n,n} \end{matrix} \right.} - \overline{\left| \begin{matrix} a_{1,1}b_{1,2} & a_{1,2}b_{1,1} \ldots a_{1,n}b_{1,n} \\ \vdots & \\ a_{n,1}b_{n,2} & a_{n,2}b_{n,1} \ldots a_{n,n}b_{n,n} \end{matrix} \right.} + \ldots$$

If one of the factors of PQ, as P, be a conterminant, then it is manifest by the same sort of reasoning that the products of the 1st, 2nd, ...nth, terms of P with the corresponding ones of Q are the same as before, but with signs alternately positive and negative; so that the leading expression in PQ will be the determinant

$$\left| \begin{matrix} a_{1,1}b_{1,1} \ldots a_{1,n}b_{1,n} \\ \vdots \\ a_{n,1}b_{n,1} \quad a_{n,n}b_{n,n} \end{matrix} \right|.$$

By interchanging the second suffixes of the b's in pairs we obtain other determinants with alternate signs; so that the product PQ is obtainable from the preceding result by putting determinants in place of the corresponding conterminants.

It follows therefore that the product of the three determinants P, Q, R, where $R = \Sigma \pm c_{1,1} \ldots c_{n,n}$, will be a series of determinants alternately positive and negative derived from the leading determinant $\Sigma \pm a_{1,1}b_{1,1}c_{1,1} \ldots a_{n,n}b_{n,n}c_{n,n}$, by interchanging the second suffixes of the b's and the c's in pairs successively. And generally, the product of an even or odd number of determinants will consist of conterminants or determinants respectively, constituted as above.

Examples.

$$\left| \begin{matrix} a & b & c \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{matrix} \right| \left| \begin{matrix} d & e & f \\ d_1 & e_1 & f_1 \\ d_2 & e_2 & f_2 \end{matrix} \right| = \overline{\left| \begin{matrix} ad & be & cf \\ a_1d_1 & b_1e_1 & c_1f_1 \\ a_2d_2 & b_2e_2 & c_2f_2 \end{matrix} \right.} - \overline{\left| \begin{matrix} ad & be & cf \\ a_1d_2 & b_1e_2 & c_1f_2 \\ a_2d_1 & b_2e_1 & c_2f_1 \end{matrix} \right.}$$

$$+ \overline{\left| \begin{matrix} ad_1 & be_1 & cf_1 \\ a_1d_2 & b_1e_2 & c_1f_2 \\ a_2d & b_2e & c_2f \end{matrix} \right.} - \overline{\left| \begin{matrix} ad_1 & be_1 & cf_1 \\ a_1d & b_1e & c_1f \\ a_2d_2 & b_2e_2 & c_2f_2 \end{matrix} \right.}$$

$$+ \overline{\left| \begin{matrix} ad_2 & be_2 & cf_2 \\ a_1d & b_1e & c_1f \\ a_2d_1 & b_2e_1 & c_2f_1 \end{matrix} \right.} - \overline{\left| \begin{matrix} ad_2 & be_2 & cf_2 \\ a_1d_1 & b_1e_1 & c_1f_1 \\ a_2d & b_2e & c_2f \end{matrix} \right.}.$$

$$\begin{vmatrix} a & b & c \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}^2 = \overline{\begin{vmatrix} a^2 & b^2 & c^2 \\ a_1^2 & b_1^2 & c_1^2 \\ a_2^2 & b_2^2 & c_2^2 \end{vmatrix}} + 2\,\overline{\begin{vmatrix} aa_1 & bb_1 & cc_1 \\ aa_2 & bb_2 & cc_2 \\ a_1a_2 & b_1b_2 & c_1c_2 \end{vmatrix}}$$

$$- \overline{\begin{vmatrix} a^2 & b^2 & c^2 \\ a_1a_2 & b_1b_2 & c_1c_2 \\ a_1a_2 & b_1b_2 & c_1c_2 \end{vmatrix}} - \overline{\begin{vmatrix} a_1^2 & b_1^2 & c_1^2 \\ aa_2 & bb_2 & cc_2 \\ aa_2 & bb_2 & cc_2 \end{vmatrix}} - \overline{\begin{vmatrix} a_2^2 & b_2^2 & c_2^2 \\ aa_1 & bb_1 & cc_1 \\ aa_1 & bb_1 & cc_1 \end{vmatrix}}$$

$$= \Sigma a_1^4 a_2^2 - 2\Sigma a_1^4 a_2 a_3 + 2\Sigma a_1^3 a_2^3 a_3 - 6a_1^2 a_2^2 a_3^2.$$

$$\{(b-a)(c-a)(c-b)\}^2$$

$$= \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}^2 = \overline{\begin{vmatrix} 1 & a^2 & a^4 \\ 1 & b^2 & b^4 \\ 1 & c^2 & c^4 \end{vmatrix}} + 2\,\overline{\begin{vmatrix} 1 & ab & a^2b^2 \\ 1 & ac & a^2c^2 \\ 1 & bc & b^2c^2 \end{vmatrix}} - \overline{\begin{vmatrix} 1 & a^2 & a^4 \\ 1 & bc & b^2c^2 \\ 1 & bc & b^2c^2 \end{vmatrix}}$$

$$- \overline{\begin{vmatrix} 1 & b^2 & b^4 \\ 1 & ac & a^2c^2 \\ 1 & ac & a^2c^2 \end{vmatrix}} - \overline{\begin{vmatrix} 1 & c^2 & c^4 \\ 1 & ab & a^2b^2 \\ 1 & ab & a^2b^2 \end{vmatrix}}$$

$$= \Sigma a^4b^2 - 2\Sigma a^4bc - 2\Sigma a^3b^3 + 2\Sigma a^3b^2c - 6a^2b^2c^2.$$

Everton Vicarage, near St. Neots,
October, **1865.**

THEOREM RELATING TO THE FOUR CONICS WHICH TOUCH THE SAME TWO LINES AND PASS THROUGH THE SAME FOUR POINTS.

By Professor CAYLEY.

THE sides of the triangle formed by the given points meet one of the given lines in three points, say P, Q, R; and on this same line we have four points of contact, say A_1, A_2, A_3, A_4; any two pairs, say A_1, A_2; A_3, A_4, form with a properly selected pair, say Q, R, out of the above-mentioned three points, an involution; and we have thus the three involutions

$$(A_1, A_2;\ A_3, A_4;\ Q, R),$$
$$(A_1, A_3;\ A_4, A_2;\ R, P),$$
$$(A_1, A_4;\ A_2, A_3;\ P, Q).$$

To prove this, let $x=0$, $y=0$ be the equations of the given lines, and take for the equations of the sides of the triangle formed by the given points

$$\begin{aligned} bx + ay - ab &= 0, \\ b'x + a'y - a'b' &= 0, \\ b''x + a''y - a''b'' &= 0. \end{aligned}$$

The equation of any one of the four conics may be written

$$\frac{Lab}{bx+ay-ab} + \frac{L'a'b'}{b'x+a'y-a'b'} + \frac{L''a''b''}{b''x+a''y-a''b''} = 0,$$

and if this touches the axis of x, say at the point $x=\alpha$, then we must have

$$\frac{La}{x-a} + \frac{L'a'}{x-a'} + \frac{L''a''}{x-a''} = \frac{-K(x-\alpha)^2}{(x-a)(x-a')(x-a'')}.$$

Or, assuming as we may do, $K = -(a'-a'')(a''-a)(a-a')$, this gives

$$\begin{aligned} La &= (a-\alpha)^2(a'-a''), \\ L'a' &= (a'-\alpha)^2(a''-a), \\ L''a'' &= (a''-\alpha)^2(a-a'). \end{aligned}$$

But in the same manner, if the conic touch the axis of y, say at the point $y=\beta$, we have

$$\begin{aligned} Lb &= (b-\beta)^2(b'-b''), \\ L'b' &= (b'-\beta)^2(b''-b), \\ L''b'' &= (b''-\beta)^2(b-b'). \end{aligned}$$

And thence

$$b(a-\alpha)^2(a'-a'') : b'(a'-\alpha)^2(a''-a) : b''(a''-\alpha)^2(a-a')$$
$$= a(b-\beta)^2(b'-b'') : a'(b'-\beta)^2(b''-b) : a''(b''-\beta)^2(b-b').$$

Or putting
$$\begin{aligned} P &= ab\ (a'-a'')(b'-b''), \\ P' &= a'b'\ (a''-a)(b''-b), \\ P'' &= a''b''(a-a')(b-b'), \end{aligned}$$

we have
$$(a-\alpha)^2\frac{P}{a^2} : (a'-\alpha)^2\frac{P'}{a'^2} : (a''-\alpha)^2\frac{P''}{a''^2}$$
$$= (b-\beta)^2(b'-b'')^2 : (b'-\beta)^2(b''-b)^2 : (b''-\beta)^2(b-b')^2;$$

and thence

$$(a-\alpha)\frac{\sqrt{(P)}}{a} : (a'-\alpha)\frac{\sqrt{(P')}}{a'} : (a''-\alpha)\frac{\sqrt{(P'')}}{a''}$$
$$= (b-\beta)(b'-b'') : (b'-\beta)(b''-b) : (b''-\beta)(b-b'),$$

which gives

$$(a-\alpha)\frac{\sqrt{(P)}}{a}+(a'-\alpha)\frac{\sqrt{(P')}}{a'}+(a''-\alpha)\frac{\sqrt{(P'')}}{a''}=0,$$

and we have in like manner

$$(b-\beta)\frac{\sqrt{(P)}}{b}+(b'-\beta)\frac{\sqrt{(P')}}{b'}+(b''-\beta)\frac{\sqrt{(P'')}}{b''}=0,$$

but the first of these equations is alone required for the present purpose. Putting for shortness

$$P=a^2X,\quad P'=a'^2X',\quad P''=a''^2X'',$$

the equation is

$$(a-\alpha)\sqrt{(X)}+(a'-\alpha)\sqrt{(X')}+(a''-\alpha)\sqrt{(X'')},$$

and by attributing the signs + and − to the radicals, we have corresponding to the four conics, the equations

$$\begin{aligned}(a-\alpha_1)\sqrt{(X)}+(a'-\alpha_1)\sqrt{(X')}+(a''-\alpha_1)\sqrt{(X'')}&=0,\\ -(a-\alpha_2)\sqrt{(X)}+(a'-\alpha_2)\sqrt{(X')}+(a''-\alpha_2)\sqrt{(X'')}&=0,\\ (a-\alpha_3)\sqrt{(X)}-(a'-\alpha_3)\sqrt{(X')}+(a''-\alpha_3)\sqrt{(X'')}&=0,\\ (a-\alpha_4)\sqrt{(X)}+(a'-\alpha_4)\sqrt{(X')}-(a''-\alpha_4)\sqrt{(X'')}&=0,\end{aligned}$$

where α_1, α_2, α_3, α_4 are the values of α for the four conics respectively.

Eliminating a'' we obtain the system of three equations

$$\begin{aligned}&(2a-\alpha_1-\alpha_2)\sqrt{(X)}+(\alpha_2-\alpha_1)\sqrt{(X')}+(\alpha_2-\alpha_1)\sqrt{(X'')}=0,\\ &(\alpha_3-\alpha_1)\sqrt{(X)}+(2a'-\alpha_1-\alpha_3)\sqrt{(X')}+(\alpha_3-\alpha_1)\sqrt{(X'')}=0,\\ &(\alpha_1+\alpha_2-\alpha_3-\alpha_4)\sqrt{(X)}+(\alpha_1+\alpha_3-\alpha_2-\alpha_4)\sqrt{(X')}\\ &\qquad+(\alpha_1+\alpha_4-\alpha_2-\alpha_3)\sqrt{(X'')}=0,\end{aligned}$$

and then eliminating the radicals we have

$$\begin{vmatrix}2a-\alpha_1-\alpha_2, & \alpha_2-\alpha_1, & \alpha_2-\alpha_1\\ \alpha_3-\alpha_1, & 2a'-\alpha_1-\alpha_3, & \alpha_3-\alpha_1\\ \alpha_1+\alpha_2-\alpha_3-\alpha_4, & \alpha_1+\alpha_3-\alpha_2-\alpha_4, & \alpha_1+\alpha_4-\alpha_2-\alpha_3\end{vmatrix}=0,$$

which is in fact

$$-4\begin{vmatrix}1, & a+a', & aa'\\ 1, & \alpha_1+\alpha_4, & \alpha_1\alpha_4\\ 1, & \alpha_2+\alpha_3, & \alpha_2\alpha_3\end{vmatrix}=0,$$

as may be verified by actual expansion; the transformation of the determinant is a peculiar one.

The foregoing result was originally obtained as follows, viz. writing for a moment

$$a\sqrt{(X)}+a'\sqrt{(X')}+a''\sqrt{(X'')}=\Theta,$$
$$\sqrt{(X)}+\sqrt{(X')}+\sqrt{(X'')}=\Phi,$$

the four equations are

$$\begin{aligned}\Theta-\alpha_1\phi&=0,\\ \Theta-\alpha_2\phi&=2(a-\alpha_2)\sqrt{(X)},\\ \Theta-\alpha_3\phi&=2(a'-\alpha_3)\sqrt{(X')},\\ \Theta-\alpha_4\phi&=2(a''-\alpha_4)\sqrt{(X'')};\end{aligned}$$

these give

$$\begin{aligned}(\alpha_1-\alpha_2)\phi&=2(a-\alpha_2)\sqrt{(X)},\\ (\alpha_1-\alpha_3)\phi&=2(a'-\alpha_3)\sqrt{(X')},\\ (\alpha_1-\alpha_4)\phi&=2(a''-\alpha_4)\sqrt{(X'')}.\end{aligned}$$

From the last equation we have

$$\begin{aligned}(\alpha_1-\alpha_4)\Phi&=2\{\Theta-a\sqrt{(X)}-a'\sqrt{(X')}\}-2\alpha_4\{\phi-\sqrt{(X)}-\sqrt{(X')}\}\\ &=2(\alpha_1-\alpha_4)\phi-2(a-\alpha_4)\sqrt{(X)}-2(a'-\alpha_4)\sqrt{(X')};\end{aligned}$$

that is

$$(\alpha_1-\alpha_4)\phi-2(a-\alpha_4)\sqrt{(X)}-2(a'-\alpha_4)\sqrt{(X')}=0;$$

or substituting for $\sqrt{(X)}$, $\sqrt{(X')}$ their values in terms of ϕ, we find

$$\alpha_1-\alpha_4-\frac{(a-\alpha_4)(\alpha_1-\alpha_2)}{a-\alpha_2}-\frac{(a'-\alpha_4)(\alpha_1-\alpha_3)}{a'-\alpha_3}=0,$$

which may be written

$$\alpha_1-\alpha_4-(\alpha_1-\alpha_2)\left(1+\frac{\alpha_2-\alpha_4}{a-\alpha_2}\right)-(\alpha_1-\alpha_3)\left(1+\frac{\alpha_3-\alpha_4}{a'-\alpha_3}\right)=0,$$

that is

$$\alpha_2+\alpha_3-\alpha_1-\alpha_4+\frac{(\alpha_2-\alpha_1)(\alpha_2-\alpha_4)}{a-\alpha_2}+\frac{(\alpha_3-\alpha_1)(\alpha_3-\alpha_4)}{a'-\alpha_3}=0.$$

Or again

$$(\alpha_2-\alpha_1)\left(1+\frac{\alpha_2-\alpha_4}{a-\alpha_2}\right)+(\alpha_3-\alpha_4)\left(1+\frac{\alpha_3-\alpha_1}{a'-\alpha_3}\right)=0,$$

that is

$$(\alpha_2-\alpha_1)\frac{a-\alpha_4}{a-\alpha_2}+(\alpha_3-\alpha_4)\frac{a'-\alpha_1}{a'-\alpha_3}=0.$$

Or finally

$$(\alpha_2-\alpha_1)(a-\alpha_4)(a'-\alpha_3)+(\alpha_3-\alpha_4)(a-\alpha_2)(a'-\alpha_1)=0,$$

which is a known form of the relation

$$\begin{vmatrix} 1, & a+a', & aa' \\ 1, & \alpha_1+\alpha_4, & \alpha_1\alpha_4 \\ 1, & \alpha_2+\alpha_3, & \alpha_2\alpha_3 \end{vmatrix} = 0,$$

which gives the involution of the quantities a, a'; α_1, α_4; α_2, α_3.

We have in like manner

$$\begin{vmatrix} 1, & a'+a'', & a'a'' \\ 1, & \alpha_1+\alpha_2, & \alpha_1\alpha_2 \\ 1, & \alpha_3+\alpha_4, & \alpha_3\alpha_4 \end{vmatrix} = 0,$$

and

$$\begin{vmatrix} 1, & a''+a, & a''a \\ 1, & \alpha_1+\alpha_3, & \alpha_1\alpha_3 \\ 1, & \alpha_2+\alpha_4, & \alpha_2\alpha_4 \end{vmatrix} = 0,$$

which give the involutions of the systems a', a''; α_1, α_2; α_3, α_4 and a'', a; α_1, α_3; α_2, α_4 respectively.

It may be remarked that the equation of the conic passing through the three points and touching the axis of x in the point $x=\alpha$ is

$$\frac{(a-\alpha)^2(a'-a'')\,b}{bx+ay-ab} + \frac{(a'-\alpha)^2(a''-a)\,b'}{b'x+a'y-a'b'} + \frac{(a''-\alpha)^2(a-a')\,b''}{b''x+a''y-a''b''} = 0,$$

and when this meets the axis of y we have

$$\frac{\frac{b}{a}(a-\alpha)^2(a'-a'')}{y-b} + \frac{\frac{b'}{a'}(a'-\alpha)^2(a''-a)}{y-b'} + \frac{\frac{b''}{a''}(a''-\alpha)^2(a-a')}{y-b''} = 0.$$

Hence, if this touches the axis of y in the point $y=\beta$, the left-hand side must be

$$= \frac{\left[\frac{b}{a}(a-\alpha)^2(a'-a'')+\frac{b'}{a'}(a'-\alpha)^2(a''-a)+\frac{b''}{a''}(a''-\alpha)^2(a-a')\right](y-\beta)^2}{(y-b)(y-b')(y-b'')},$$

and equating the coefficients of $\frac{1}{y^2}$, we have

$$\frac{b^2}{a}(a-\alpha)^2(a'-a'') + \frac{b'^2}{a'}(a'-\alpha)^2(a''-a) + \frac{b''^2}{a''}(a''-\alpha)^2(a-a')$$

$$= \left[\frac{b}{a}(a-\alpha)^2(a'-a'') + \frac{b'}{a'}(a'-\alpha)^2(a''-a) \right.$$

$$\left. + \frac{b''}{a''}(a''-\alpha)^2(a-a')\right](b+b'+b''-2\beta),$$

or what is the same thing,

$$\frac{b(b'+b'')}{a}(a-\alpha)^2(a'-a'')+\frac{b'(b''+b)}{a'}(a'-\alpha)^2(a''-a)$$
$$+\frac{b''(b+b')}{a''}(a''-\alpha)^2(a-a')$$

$$=2\beta\left[\frac{b}{a}(a-\alpha)^2(a'-a'')+\frac{b'}{a'}(a'-\alpha)^2(a''-a)+\frac{b''}{a''}(a''-\alpha)^2(a-a')\right],$$

which gives β in terms of α, that is $\beta_1, \beta_2, \beta_3, \beta_4$ in terms of $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ respectively.

Cambridge, 30 *November*, 1863.

A CONSEQUENCE OF MR. CAYLEY'S THEORY OF SKEW DETERMINANTS, CONCERNING THE DISPLACEMENT OF A RIGID SYSTEM OF AN EVEN NUMBER OF DIMENSIONS ABOUT A FIXED ORIGIN.

By Prof. Schläfli.

1. When we displace a rigid system of n orthogonal axes, represented in its primitive state by the identical substitution

$$\begin{vmatrix} 1, & 0, & 0, & \ldots \\ 0, & 1, & 0, & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{vmatrix},$$

about its origin to an infinitely small extent, the substitution representing its altered state in relation to the primitive axes cannot but assume the form

$$\begin{vmatrix} 1, & a, & b, & c, & \ldots \\ -a, & 1, & f, & g, & \ldots \\ -b, & -f, & 1, & h, & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{vmatrix},$$

where $a, b, \ldots$ denote infinitesimals of the first order, and squares and products of them are neglected; for else the new system could not fulfil the orthogonal conditions. By erasing the units on the diagonal we should obtain the coefficients for the representation of the *projections of the displace-*

ment of any point of the rigid system; should we then demand their vanishing in order to find resting points, a skew symmetrical determinant ought to vanish. But this is only then generally possible when n is odd; and as in this case the first minors do not vanish, there is consequently an axis, all points of which rest immoveable during the displacement. Since such axis may be taken as a primitive one, and the corresponding dimension dropped, the question reduces to an even number of dimensions.

Let $w_1, w_2, \ldots w_{2n}$ be the coordinates of a point,

$$p_r = \sum_{s=1}^{s=2n} \binom{r}{s} w_s \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \;(1),$$

the projections of its infinitesimal displacement, where

$$\binom{r}{r} = 0, \;\; \binom{r}{s} = -\binom{s}{r},$$

and $\quad K = \{1234\ldots(2n-1)\,(2n)\} = \Sigma \binom{1}{2}\binom{3}{4}\ldots\binom{2n-1}{2n},$

the *Tfaffian* of the $n(2n-1)$ infinitesimal elements where each term appears but once and shall bear the sign $+$, provided that it be written with a *positive* permutation. Again, put

$$dK = \begin{bmatrix}1\\2\end{bmatrix} d\binom{1}{2} + \ldots, \;\; d\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}13\\24\end{bmatrix} d\binom{3}{4} + \ldots,$$

so that $\quad \begin{bmatrix}13\\24\end{bmatrix} = \begin{bmatrix}14\\32\end{bmatrix} = \begin{bmatrix}12\\43\end{bmatrix} = \{5678\ldots(2n)\},$

and further on put

$$\begin{bmatrix}1\\2\end{bmatrix}\begin{bmatrix}3\\4\end{bmatrix} + \begin{bmatrix}1\\3\end{bmatrix}\begin{bmatrix}4\\2\end{bmatrix} + \begin{bmatrix}1\\4\end{bmatrix}\begin{bmatrix}2\\3\end{bmatrix} = [1234],$$

$$\begin{bmatrix}1\\2\end{bmatrix}[3456] + \begin{bmatrix}1\\3\end{bmatrix}[4256] + \begin{bmatrix}1\\4\end{bmatrix}[2356]$$

$$+ \begin{bmatrix}1\\5\end{bmatrix}[3246] + \begin{bmatrix}1\\6\end{bmatrix}[2345] = [123456], \text{ \&c.}$$

We then have the inverse system

$$Kw_s = \sum_{r=1}^{r=2n} \begin{bmatrix}r\\s\end{bmatrix} p_r, \;\; (s = 1, 2, \ldots 2n) \;\ldots\ldots\ldots\; (2),$$

where $\quad \begin{bmatrix}s\\s\end{bmatrix} = 0, \;\; \begin{bmatrix}r\\s\end{bmatrix} = -\begin{bmatrix}s\\r\end{bmatrix}.$

The systems (1) and (2) give respectively

$$\sum_{r=4}^{r=2n} \begin{bmatrix} 13 \\ 2r \end{bmatrix} p_r + \begin{bmatrix} 2 \\ 3 \end{bmatrix} w_1 + \begin{bmatrix} 3 \\ 1 \end{bmatrix} w_2 + \begin{bmatrix} 1 \\ 2 \end{bmatrix} w_3 = 0,$$

$$\Sigma\,[123r]\,p_r + K\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix} w_1 + \begin{bmatrix} 3 \\ 1 \end{bmatrix} w_2 + \begin{bmatrix} 1 \\ 2 \end{bmatrix} w_3\right) = 0,$$

whence $$[1234] = K \begin{bmatrix} 13 \\ 24 \end{bmatrix}.$$

In a similar manner, by deriving from both systems (1) and (2), two relations between $w_1, w_2, w_3, w_4, w_5, p_6, p_7, \ldots p_{2n}$, and comparing them we find, in virtue of the result just obtained, $[123456] = K^2 \begin{bmatrix} 135 \\ 246 \end{bmatrix}$, and so proceeding we could successively prove

$$[1234\ldots(2r)] = K^{r-1}\,\{(2r+1)\,(2r+2)\ldots(2n)\} \;\ldots.(3).$$

Though the angular digression $\sqrt{(\Sigma p^2)} : \sqrt{(\Sigma w^2)}$ cannot vanish, it must yet have a greatest and a least values. Let such one be $\sqrt{(w)} = \sqrt{\left(\frac{K}{j}\right)}$. The several derivatives then of $j\Sigma p^2 - K\Sigma w^2$ taken with regard either to $w_1, w_2 \ldots$ or to $p_1, p_2, \ldots$ must vanish. Hence the two equivalent systems of conditions

$$j\left\{\binom{2}{1} p_2 + \binom{3}{2} p_3 \ldots + \binom{2n}{1} p_{2n}\right\} = Kw_1 \;\ldots\ldots (4),$$

$$jp_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} w_2 + \begin{bmatrix} 1 \\ 3 \end{bmatrix} w_3 \ldots + \begin{bmatrix} 1 \\ 2n \end{bmatrix} w_{2n} \;\ldots\ldots.. (5).$$

On putting for brevity $j\binom{1}{2} - \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \lambda_{12}$ (whence $\lambda_{11} = 0$, $\lambda_{12} = -\lambda_{21}$) these conditions, on account of (1), (2), become $\Sigma_s \lambda_{rs} p_s = 0$, $\Sigma_s \lambda_{rs} w_s = 0$, two skew symmetrical equation-systems which are identical with one another in respect to their constant elements. Since the determinant of such a system is the square of the Tfaffian $\lambda\,\{123\ldots(2n)\}$, that Tfaffian must vanish, and then the first minors of the determinant will vanish too so as not to determine the ratios of $w_1, w_2, \ldots$, nor those of $p_1, \ldots$ either. That is to say, the locus of all those points, the angular digression of which about the origin is a maximum, cannot be a line parting from origin, but is a

plane passing through the origin. And the displacement itself lies in this plane, for its projections satisfy the same system of conditions as the coordinates of the displaced point do. The displacement is of course perpendicular to the radius vector of the point, for the system (1) gives generally $\Sigma pw = 0$.

By the help of the relation (3) the last said Tfaffian's vanishing furnishes for the square ω of the greatest angular digression, the equation

$$\omega^n - \Sigma (12)^2 . \omega^{n-1} + \Sigma (1234)^2 . \omega^{n-2} - \Sigma (123456)^2 . \omega^{n-3} \ldots$$
$$+ (-1)^n K^2 = 0 \ldots\ldots\ldots\ldots (6),$$

which shows, by the sums of squares and alternation of signs, that its roots, when real, are positive. We shall soon see that they are all real.

If we distinguish a second solution of (1), (4) by an accent, the system (1) gives $\Sigma pw' = -\Sigma p'w$, the system (2) $K\Sigma ww' = j'\Sigma pp' = j\Sigma pp'$ on account of (5), and the system (5) gives $j\Sigma pw' + j'\Sigma p'w = 0$. Hence

$$(j - j')\, \Sigma pp' = 0, \quad (j - j')\, \Sigma pw' = 0,$$

and unless the two roots j, j' of equation (6) be equal, $\Sigma pp' = 0$, $\Sigma ww' = 0$, $\Sigma pw' = 0$, $\Sigma p'w = 0$, and of course also

$$\Sigma\, (\alpha p + \beta w)\, (\gamma p' + \delta w') = 0,$$

which shows that every *ray* (I shall call so any line parting from the origin) in the plane (j) forms a right angle with every ray in the plane (j'). A pair of equal roots in (6) would cause all the first minor Tfaffians such as $\lambda\, \{345 \ldots (2n)\}$ to have a common root with (6). But, for the present, we will not enter upon so great a limitation which imports instead of a plane a linear continuum of four dimensions where all angular digressions are equal. If j, j' were imaginary and conjugate, there were also in both conjugate planes any pair of conjugate rays which ought to satisfy $\Sigma ww' = 0$, which is impossible from known reasons. Therefore the n planes furnished by equation (6) are all of them real and orthogonal each to each, and on taking in each of them at random two axes forming a right angle at the origin we shall have the $2n$ axes of an orthogonal system of coordinates. These axes being adopted so that the *principal* plane (j) is defined by $w_3 = 0$, $w_4 = 0$, ... $w_{2n} = 0$, the system (5) gives $\lambda_{12} = 0$, and since $p_3, p_4, \ldots p_{2n}$ must also vanish, but their vanishing must not determine the ratio $w_1 : w_2$, all the

elements $\binom{r}{s}$ where $s=3, 4, \dots 2n$ are forced to vanish. Then $K=\binom{1}{2}\begin{bmatrix}1\\2\end{bmatrix}$, $\omega=\binom{1}{2}^2$. When so again in the plane (j') all coordinates but w_3, w_4 vanish, in the plane (j'') all but w_5, w_6, and so on; then only the elements $\binom{1}{2}$, $\binom{3}{4}$, $\dots\binom{2n-1}{2n}$ can be different from zero, and are at the same time values of $\sqrt{(\omega)}$ relative to each of the n principal planes. On putting $\binom{2r-1}{2r}=a_r$, the system (1) becomes $p_1=a_1w_2$, $p_2=-a_1w_1$, $p_3=a_2w_4$, $p_4=-a_2w_3$, $\dots p_{2n-1}=a_nw_{2n}$, $p_{2n}=-a_nw_{2n-1}$. Call the origin O, any given point P, its projections on the principal planes P_1, P_2, $\dots P_n$, the displacements of these n points taken as parts of the rigid system P_1Q_1, $\dots P_nQ_n$; then the directions OP_1, P_1Q_1, OP_2, P_2Q_2, $\dots OP_n$, P_nQ_n make up an orthogonal system, which we may transport so as to have P for origin. The directions of OP_1, OP_2, $\dots OP_n$, then determine a linear continuum of n dimensions (∞^n) passing through P, the directions P_1Q_1, $\dots P_nQ_n$ another ∞^n complementary and orthogonal to the former. It is altogether in this latter ∞^n that the displacement of P lies and has therein the infinitesimal lengths P_1Q_1, P_2Q_2, $\dots$ for its projections.

The short conclusion from this discussion is: an infinitesimal movement of a rigid system of an even number of dimensions about a fixed origin can be regarded as a passage from

$$\begin{vmatrix} 1, & 0, & 0, & \dots \\ 0, & 1, & 0, & \dots \\ 0, & 0, & 1, & \dots \\ \dots & \dots & \dots & \dots \end{vmatrix} \text{ to } \begin{vmatrix} 1, & a, & 0, & 0, & \dots \\ -a, & 1, & 0, & 0, & \dots \\ 0, & 0, & 1, & b, & \dots \\ 0, & 0, & -b, & 1, & \dots \\ \dots & \dots & \dots & \dots & \dots \end{vmatrix},$$

where a, b, $\dots$ denote infinitesimal angular movements. Hence it appears at once that an odd number of dimensions gives rise to a passage from the identical substitution to

$$\begin{vmatrix} 1, & 0, & 0, & 0, & 0, & \dots \\ 0, & 1, & a, & 0, & 0, & \dots \\ 0, & -a, & 1, & 0, & 0, & \dots \\ 0, & 0, & 0, & 1, & b, & \dots \\ 0, & 0, & 0, & -b, & 1, & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{vmatrix}.$$

2. We go on to consider a *finite* displacement about the fixed origin, and try if we can represent it by a passage from the identical system to

$$\begin{vmatrix} \lambda, & \mu, & 0, & 0, \ldots \\ -\mu, & \lambda, & 0, & 0, \ldots \\ 0, & 0, & \lambda', & \mu', \ldots \\ 0, & 0, & -\mu', & \lambda', \ldots \\ \ldots & \ldots & \ldots & \ldots \end{vmatrix},$$

where $\lambda^2+\mu^2=1$, $\lambda'^2+\mu'^2=1$, &c. For $2n$ dimensions the determination of the first principal plane (12) requires $4(n-1)$ elements to be known, that of the second (34), $4(n-2)$ ones, &c. The position of all the n principal planes accordingly counts for $2n(n-1)$ unknown quantities, which number is the same as that of the independent elements of an orthogonal substitution. Both from what we have seen with the infinitesimal displacement and from this enumeration it is very probable that the representation in question is always possible.

Let $w'=\binom{1}{1}w+\binom{1}{2}x+\binom{1}{3}y\ldots+\binom{1}{2n}z$, $x'=\binom{2}{1}w+\ldots$, &c.

be an orthogonal substitution, and

$$w_{,}=\binom{1}{1}w+\binom{2}{1}x+\binom{3}{1}y\ldots+\binom{2n}{1}z, \text{ \&c.}$$

the inverse one, and call the former S, the latter S^{-1}, and put, if it be wanted,

$$(w'', x'', \ldots)=S^2(w, x, \ldots),\ \ (w'''\ldots)=S^3(w\ldots), \text{ \&c.},$$
$$(w_{,,}\ldots)=S^{-2}(w\ldots), \text{ \&c.}$$

It is plain that

$$w'^2+x'^2\ldots+z'^2=w^2+x^2+\ldots+z^2.$$

If we put $\quad ww'+xx'\ldots+zz'=s(w^2+x^2\ldots+z^2)\ldots\ldots\ldots(7),$

s will be the cosine of the angle formed by the two rays $(w\ldots)$, $(w'\ldots)$ at the origin (and $\Sigma ww'=\Sigma ww_{,}$ is identically true). When the given substitution is real and we confine our regard to real rays, such angle will have a real absolute maximum and also such a minimum. Differentiating then equation (7) in respect to $w, x, \ldots z$, we obtain the conditions

$$w'+w_{,}-2sw=0,\ x'+x_{,}-2sx=0,\ \ldots,\ z'+z_{,}-2sz=0\ldots(8),$$

multiplying them by $\binom{1}{1}$, $\binom{1}{2}$, ... $\binom{1}{2n}$ and adding we get

$w''+w-2sw'=0$, and so on (system 9). It is proper to remark that (8) and (9) subsist with the same value of s. The system (8), after elimination of the $2n-1$ ratios of $w, x\ldots$, gives a symmetrical determinant of degree $2n$ in s vanishing. Should at the same time its first minors (belonging to one and the same line) determine the said ratios, those of $(w', x'\ldots)$ would be the same, so that $\frac{w'}{w}=\frac{x'}{x}\ldots=\frac{z'}{z}$, which leads to $w^2+x^2\ldots+z^2=0$, inconsistent with the reality of the ray of greatest angular digression. Therefore all first minors must vanish at the same time as the determinant, s must be a double root, the system (8) represents a principal plane (s), which contains along with the ray $(w\ldots)$, also $\ldots(w_{\prime\prime}\ldots)$, $(w_{\prime}\ldots)$, $(w\ldots)$, $(w'\ldots)$, $(w''\ldots)$, $(w'''\ldots)$, $\ldots$ forming in this succession equal angles each with that immediately following. Moreover as it is analytically impossible that the case of absolute maximum be intrinsically different from any other solution of (8), all values of s must be double roots, say, that determinant $(s, 1)^{2n}$ is a perfect square, call V^2, which can be proved directly too. For since only the diagonal terms $2\binom{1}{1}-2s$, $2\binom{2}{2}-2s$, $\ldots$ contain s, the other being of the form $\binom{1}{2}+\binom{2}{1}$, the coefficient of s^{2n} is 2^{2n}. The determinant got by eliminating $w, x, \ldots$ from the system (9) must be the same, up to a numerical factor. Let $w''=a_{11}w+a_{12}x+a_{13}y+\ldots$, so that

$$a_{rs}=\binom{r}{1}\binom{1}{s}+\binom{r}{2}\binom{2}{s}+\binom{r}{3}\binom{3}{s}\ldots+\binom{r}{2n}\binom{2n}{s},$$

then the diagonal elements of the determinant (9) are of the form $a_{11}+1-2\binom{1}{1}s$, the second of the first line is $a_{12}-2\binom{1}{2}s$. Hence the coefficient of s^{2n} is $2^{2n}\binom{123\ldots2n}{123\ldots2n}=2^{2n}$ as before. Both determinants are therefore identical. Now assume $2s=t+\frac{1}{t}$ and call $T(t)$ the determinant got by eliminating $w, x, \ldots$ from the $2n$ equations such as $w'-tw=0$, where the first line is $\binom{1}{1}-t$, $\binom{1}{2}$, $\binom{1}{3}\ldots\binom{1}{2n}$; then replace t by $\frac{1}{t}$, turn the determinant $T\left(\frac{1}{t}\right)$ about its diagonal so as

to change columns into lines and multiply it by the former $T'(t)$ line by line. There will arise the determinant (9). But $T(t)$ is reciprocal in respect to t, and its extreme terms are $t^{2n}+1$. Divided by t^n it therefore becomes $(2^n, \ldots \between s, 1)^n$ equal to $t^n . T\left(\frac{1}{t}\right)$. In consequence thereof $V = t^{-n} T(t)$ is right. The geometrical meaning of the set of equations whose determinant is T will soon be explained.

Because the absolute maximum and minimum of the angle of digression are real, the equation $V=0$ has at least two real roots between -1 and 1. Let $\bar{s}$ be a second root and $(\bar{w}, \bar{x}, \ldots)$ a point in the principal plane $(\bar{s})$, multiply the equations (8) respectively by $\bar{w}, \bar{x}, \ldots$ and add; you will obtain $(\bar{s}-s)(w\bar{w}+x\bar{x}\ldots+z\bar{z})=0$, whence the principal planes are orthogonal to each other. From $\Sigma w\bar{w}=0$ it is easily seen that all the roots of $V=0$ are real. We are still to prove however, that they are comprised between -1 and 1. Placing four axes of the system of coordinates in those two principal planes first said, and dropping the corresponding dimensions we come back to $2n-4$ dimensions, and here $V=0$ will have again two real roots between -1 and 1, and so on. Or rather so; whenever s is real, there are real rays $(w\ldots)$ and $(w'\ldots)$ satisfying (8), and the angle between them is of course real.

The ray $(w'-\alpha w, x'-\alpha x, \ldots z'-\alpha z)$ describes the principal plane (s), as while α varies; $\Sigma(w'-\alpha w)^2 = (1-2\alpha s+\alpha^2)\Sigma w^2$ cannot vanish, but α be t or $\frac{1}{t}$, whenever the ray $(w\ldots)$ is real; and we propose to carry on that ray, so as to pass through a circular point at infinity. In general, that ray after the displacement of the rigid system takes the position $(w''-\alpha w', x''-\alpha x', \ldots)$, and since the equation $w''+w-2sw'=0$ is equivalent with $w''-tw' = \frac{1}{t}(w'-tw)$, we have

$$(w''-\alpha w')-t(w'-\alpha w) = \left(\frac{1}{t}-\alpha\right)\times(w'-tw),$$

which vanishes in the particular case of $\alpha = \frac{1}{t}$. Replacing then $w'-\alpha w$ by w, &c., we have for one of those two rays in the principal plane (s) which pass through the circular points at infinity, the set of conditions

$$w'-tw=0,\ x'-tx=0, \ldots,\ z'-tz=0,$$

and in consequence thereof and of (8), this other set

$$w_{,} - \frac{1}{t} w = 0, \quad x_{,} - \frac{1}{t} x = 0, \ldots, \quad z_{,} - \frac{1}{t} z = 0.$$

From whichever set we may eliminate $w, x, \ldots z$, we arrive at $T = 0$. So it is also geometrically evident that $V = T$.

3. It remains to speak of the case when $V = 0$ has two equal roots. By adapting $2n - 4$ axes of coordinates to the principal planes relating to the rest of roots, and then dropping the corresponding dimensions, we may reduce the question to four dimensions. Here is

$$V = s^2 - \tfrac{1}{2}\Sigma\binom{1}{1}s + \tfrac{1}{2}\left\{\Sigma\binom{12}{12} - 1\right\},$$

whence $4(s' - s)^2 = \Sigma\left\{\binom{1}{1} - \binom{2}{2}\right\}^2 + 2\Sigma\left\{\binom{1}{2} + \binom{2}{1}\right\}^2,$

The reality of the orthogonal elements then requires

$$\binom{1}{1} = \binom{2}{2} = \binom{3}{3} = \binom{4}{4}, \quad \binom{2}{1} = -\binom{1}{2}, \text{ \&c.}$$

that is to say, the orthogonal substitution must be skew. Moreover, the relations

$$\binom{12}{12} = \binom{34}{34}, \quad \binom{12}{13} = \binom{34}{42}, \quad \binom{12}{14} = \binom{34}{23},$$

give $\binom{3}{4}^2 = \binom{1}{2}^2, \quad \binom{3}{4}\binom{4}{2} = \binom{1}{2}\binom{1}{3}, \quad \binom{3}{4}\binom{2}{3} = \binom{1}{2}\binom{1}{4},$

and if we decide for a positive value of the Tfaffian (1234), we may write down as a substitution

$$\left\{\begin{matrix} \lambda, & \varepsilon, & \zeta, & \eta \\ -\varepsilon, & \lambda, & \eta, & -\zeta \\ -\zeta, & -\eta, & \lambda, & \varepsilon \\ -\eta, & \zeta, & -\varepsilon, & \lambda \end{matrix}\right\}.$$

The choice of the other sign was nothing else than a permutation of the indices 3, 4. Of course $\lambda^2 + \varepsilon^2 + \zeta^2 + \eta^2 = 1$, whereto we may add $\lambda^2 + \mu^2 = 1$. If $\binom{\alpha\beta\gamma\delta}{1234}$ denote an

orthogonal transformation, and a_{rs} a transformed element, then we have

$$a_{rs} = \lambda\,(\alpha_r\alpha_s + \beta_r\beta_s + \gamma_r\gamma_s + \delta_r\delta_s)$$
$$+\,\varepsilon\left\{\binom{\alpha\beta}{rs} + \binom{\gamma\delta}{rs}\right\} + \zeta\left\{\binom{\alpha\gamma}{rs} + \binom{\delta\beta}{rs}\right\} + \eta\left\{\binom{\alpha\delta}{rs} + \binom{\beta\gamma}{rs}\right\},$$

which shows that $a_{rr} = \lambda$, $a_{rs} = -a_{sr}$, $a_{12} = -a_{34}$, &c., as ought to be foreseen. In order to throw the system (a_{rs}) into the form $\begin{smallmatrix}\cdot\,\cdot & \\ & \cdot\,\cdot\end{smallmatrix}$ we impose upon it the conditions $a_{13} = 0$, $a_{14} = 0$, which cause ε, ζ, η to be proportional to their coefficients in the expression of a_{12}, since such coefficients in the three expressions of a_{12}, a_{13}, a_{14} make up an orthogonal spatial substitution, as is easily seen from combining the skew and orthogonal matrix

$$\begin{vmatrix} \alpha_1, & \beta_1, & \gamma_1, & \delta_1 \\ -\beta_1, & \alpha_1, & -\delta_1, & \gamma_1 \\ -\gamma_1, & \delta_1, & \alpha_1, & -\beta_1 \\ -\delta_1, & -\gamma_1, & \beta_1, & \alpha_1 \end{vmatrix}$$

with $\binom{\alpha\beta\gamma\delta}{1234}$ line by line. Hence

$$a_{12} = \mu,\quad \binom{\alpha\beta}{12} + \binom{\gamma\delta}{12} = \frac{\varepsilon}{\mu},\quad \binom{\alpha\gamma}{12} + \binom{\delta\beta}{12} = \frac{\zeta}{\mu},$$
$$\binom{\alpha\delta}{12} + \binom{\beta\gamma}{12} = \frac{\eta}{\mu}.$$

Did we replace the indices 1, 2 by 3, 4, the expressions on the left-hand side would not change values. There is consequently a double series of planes which contains both principal planes sought. If the first axis $(\alpha_1, \beta_1, \gamma_1, \delta_1)$ be given at pleasure, the second will be determined by

$$\mu\alpha_2 = -\varepsilon\beta_1 - \zeta\gamma_1 - \eta\delta_1,\quad \mu\beta_2 = \varepsilon\alpha_1 + \zeta\delta_1 - \eta\gamma_1,$$
$$\mu\gamma_2 = -\varepsilon\delta_1 + \zeta\alpha_1 + \eta\beta_1,\quad \mu\delta_2 = \varepsilon\gamma_1 - \zeta\beta_1 + \eta\alpha_1.$$

In this principal plane the ray

$$(\alpha_1 + i\alpha_2,\ \beta_1 + i\beta_2,\ \gamma_1 + i\gamma_2,\ \delta_1 + i\delta_2)$$

fulfils both equations

$$\zeta w + \eta x + i\mu y - \varepsilon z = 0,\quad \eta w - \zeta x + \varepsilon y + i\mu z = 0,$$

say, it lies in a fixed imaginary plane, and the conjugate ray of course in the conjugate plane. *Through every ray of the*

rigid system there passes a principal plane determined by the condition of having a line in common with each of those fixed imaginary and conjugate planes. Just as in space through every point there can be drawn a straight line such as to intersect two directrix lines, so also here through every radius vector there passes a principal plane, and but one; this radius vector runs through its appropriate principal plane, and describes a central angle constant for the whole rigid system of four dimensions.

I have not yet been able to solve the case of three equal roots of $V=0$.

4. For what concerns four dimensions in the general case of different roots, I wish to notice, that if the orthogonal substitution has taken rise from a skew matrix, the elements whereof are

$$(11)=(22)=(33)=(44)=1, \quad (12)=-(21), \text{ \&c.},$$

the determinant Δ, then on putting

$$\left|\begin{array}{l|l} (12)+(34)=R\lambda & (13)+(42)=R\mu \\ (12)-(34)=R'\lambda' & (13)-(42)=R'\mu' \end{array}\right.$$

$$\left|\begin{array}{l|l} (14)+(23)=R\nu & \lambda^2+\mu^2+\nu^2=1 \\ (14)-(23)=R'\nu' & \lambda'^2+\mu'^2+\nu'^2=1 \end{array}\right| .$$

$$\frac{R+R'}{2}=\tan\frac{\theta}{2}, \quad \frac{R-R'}{2}=\tan\frac{\theta'}{2}, \quad \text{whence } \Delta=\left(\cos\frac{\theta}{2}\cos\frac{\theta'}{2}\right)^{-2},$$

the angles of rotation will be θ, θ', and the principal plane belonging to θ will have the equations

$$\left\{\begin{array}{l} (\lambda-\lambda')\,x+(\mu-\mu')\,y+(\nu-\nu')\,z=0 \\ -(\lambda-\lambda')\,w \qquad +(\nu+\nu')\,y-(\mu+\mu')\,z=0 \end{array}\right\};$$

and a change of the signs of λ, μ', ν' will give the other principal plane.

See Mr. Cayley's Memoirs on Skew Determinants in Crelle's *Journal*, tom. XXXII., XXXVIII., L.

Supplement to Art. 3 of a consequence of Mr. Cayley's Theory of Skew Determinants.

The case of three or more equal roots of $V=0$ may be thus discussed. Let n be the number of equal roots, and the question will come back to the case of the quantic V having all its roots equal. Let it be $(1, a, b, c \ldots \between 2s, 1)^n$, and let us

begin with considering only the first of the $n-1$ conditions $a^2-b=0$, $b^2-ac=0$, $c^2-bd=0$, &c. Since

$$t^n+\frac{1}{t^n}=\sum_{\lambda=0}(-1)^\lambda\frac{n}{n-\lambda}\binom{n-\lambda}{n-2\lambda}(2s)^{n-2\lambda}=(2s)^n-n(2s)^{n-2}+\dots;$$

therefore

$$V=(2s)^n-\Sigma\binom{1}{1}.(2s)^{n-1}+\left\{\Sigma\binom{12}{12}-n\right\}(2s)^{n-2}\dots,$$

$$n^2(n-1)(a^2-b)=(n-1)\left\{\Sigma\binom{1}{1}\right\}^2$$

$$-2n\Sigma\left\{\binom{1}{1}\binom{2}{2}-\binom{1}{2}\binom{2}{1}\right\}+n.2n.$$

But in the last term $2n$ is the sum of squares of all orthogonal elements, equal to

$$\Sigma\binom{1}{1}^2+\Sigma\left\{\binom{1}{2}^2+\binom{2}{1}^2\right\}.$$

Hence,

$$n^2(n-1)(a^2-b)=(2n-1)\Sigma\binom{1}{1}^2-2\Sigma\binom{1}{1}\binom{2}{2}+2n\Sigma\binom{1}{2}\binom{2}{1}$$

$$+n\Sigma\left\{\binom{1}{2}^2+\binom{2}{1}^2\right\}=\Sigma\left\{\binom{1}{1}-\binom{2}{2}\right\}^2+n\Sigma\left\{\binom{1}{2}+\binom{2}{1}\right\}^2.$$

The supposed reality then of the orthogonal substitution, necessitates

$$\binom{1}{1}=\binom{2}{2}=\binom{3}{3}=\dots=\binom{2n}{2n},\ \text{put}=\lambda,\ \binom{2}{1}=-\binom{1}{2},\ \&c.,$$

that is to say, the substitution must be skew (we are here in the skew property, including the equality of all diagonal elements; for else every orthogonal matrix of displacement could be made skew). If so, then it appears from (8), that $V=2^n(s-\lambda)^n$, whence $s=\lambda$. Consequently the skew property is not only required by the case under consideration, but also comprises all that is wanted.

It is worth while to notice a striking property of a skew orthogonal substitution in reference to its skew elements. If (12), (1234)..., denote Tfaffians, and if $q=(t-\lambda)^2$, $\mu^2=1-\lambda^2$, then that determinant T, above whose first line was $\binom{1}{1}-t$, $\binom{1}{2}$, ... $\binom{1}{2n}$, becomes

$$T=q^n+\Sigma(12)^2.q^{n-1}+\Sigma(1234)^2.q^{n-2}+\dots+\{123\dots(2n)\}^2=(q+\mu^2)^n,$$

$\left[\text{accordingly } \Sigma\{123\ldots(2r)\}^2 = \frac{n(n-1)\ldots(n-r+1)}{1.2\ldots r}\mu^{2r}\right]$, and vanishes for $\lambda - t = i\mu$. There is more, all its first, second, $\ldots(n-1)^{\text{th}}$ minors do so. Indeed, if we keep μ at the same value as first, but suppose λ to become indeterminate, then the sum of squares of all elements of any line will be $\lambda^2 + \mu^2$, while the sum of products got by combining any two lines, continues to vanish. On dividing all elements by $\sqrt{(\lambda^2+\mu^2)}$, the new system of course becomes again orthogonal. Hence from it is readily seen that the r^{th} minor

$$\begin{bmatrix} \alpha, & \beta\ldots\delta \\ 1, & 2\ldots r \end{bmatrix} = (\lambda^2+\mu^2)^{n-r}\begin{pmatrix} \alpha, & \beta\ldots\delta \\ 1, & 2\ldots r \end{pmatrix}$$

vanishes as $r < n$, when $\lambda = i\mu$. That the n^{th} minors are allowed not to vanish, all of them may be showed by an example. The matrix

$$\begin{vmatrix} \lambda, & -\mu, & & & & \\ \mu, & \lambda, & & & & \\ & & \lambda, & -\mu, & & \\ & & \mu, & \lambda, & & \\ & & & & \lambda, & -\mu \\ & & & & \mu, & \lambda \end{vmatrix}$$

is such as is spoken of, and yet the third minor

$$\begin{pmatrix} 135 \\ 135 \end{pmatrix} = \begin{bmatrix} 246 \\ 246 \end{bmatrix} = \begin{vmatrix} \lambda & & \\ & \lambda & \\ & & \lambda \end{vmatrix}$$

does not vanish, though λ be put $= i\mu$. The possibility of transforming the skew displacement-matrix into the catenary form just written down, however, does not depend upon the truth of that those n^{th} minors (the diagonal elements being $i\mu$) *cannot* vanish all of them, and we may, without harm, anticipate this truth, since a property of such description, relating to a whole set of minors, is not altered by orthogonal transformations. We are then right to say that the matrix

$$\begin{vmatrix} i\mu, & \binom{1}{2}, & \binom{1}{3}\cdots\binom{1}{2n} \\ \binom{2}{1}, & i\mu, & \binom{2}{3}\cdots\binom{2}{2n} \\ \cdots\cdots & \cdots\cdots & \cdots\cdots \end{vmatrix},$$

among its $2n$ lines, contains no more than n independent ones, but contains so many of them.

Now let $(u, v, \ldots y, z)$ be any ray whatever,

$$u' = \lambda u + \binom{1}{2} v \ldots + \binom{1}{2n} z, \quad v' = \binom{2}{1} u + \lambda v \ldots + \binom{2}{2n} z, \text{ \&c.},$$

$\lambda^2 + \mu^2 = 1$ supposed, call such substitution S, and let

$$(u_1, v_1, \ldots z_1) = S^{-1}(u, v, \ldots z).$$

Then as for $s = \lambda$ the system (8) is satisfied, the two rays $(u \ldots)$, $(u' \ldots)$ determine a principal plane. In this we want the ray perpendicular to $(u \ldots)$. Let it be $(u' - \theta u, v' - \theta v, \ldots)$ where θ denotes an indeterminate. The condition

$$\Sigma u (u' - \theta u) = (\lambda - \theta) \Sigma u^2 = 0$$

then gives $\theta = \lambda$. Replacing $u, v, \ldots z$ by $\alpha_1, \beta_1, \ldots \zeta_1$, and $u' - \lambda u$, $v' - \lambda v, \ldots$ by $\mu\alpha_2, \mu\beta_2, \ldots \mu\zeta_2$, we have

$$\mu\alpha_2 = \binom{1}{2}\beta_1 + \binom{1}{3}\gamma_1 \ldots + \binom{1}{2n}\zeta_1,$$

$$\mu\beta_2 = \binom{2}{1}\alpha_1 + \binom{2}{3}\gamma_1 \ldots + \binom{2}{2n}\zeta_1, \ldots,$$

$$\mu\zeta_2 = \binom{2n}{1}\alpha_1 + \binom{2n}{2}\beta_1 + \binom{2n}{3}\gamma_1 \ldots + \binom{2n}{2n-1}\varepsilon_1;$$

and if $\alpha_1^2 + \beta_1^2 + \ldots + \varepsilon_1^2 + \zeta_1^2 = 1$, be supposed, then also $\alpha_2^2 + \beta_2^2 \ldots + \zeta_2^2 = 1$. Knowing thus two perpendicular directions $(\alpha_1, \beta_1, \ldots)$, $(\alpha_2, \beta_2, \ldots)$ in the principal plane, if we put

$$u = \mu(\alpha_2 + i\alpha_1), \quad v = \mu(\beta_2 + i\beta_1), \quad \ldots, \quad z = \mu(\zeta_2 + i\zeta_1),$$

we know in the same plane also a ray which passes through a circular point at infinity, viz.,

$$u = i\mu\alpha_1 + \binom{1}{2}\beta_1 + \binom{1}{3}\gamma_1 \ldots + \binom{1}{2n}\zeta_1,$$

$$v = \binom{2}{1}\alpha_1 + i\mu\beta_1 + \binom{2}{3}\gamma_1 \ldots + \binom{2}{2n}\zeta_1, \text{ \&c.};$$

and in virtue of what has above been said of the matrix of coefficients, this ray fulfils n independent conditions, and no more, such as $i\mu u + \binom{2}{1} v + \binom{3}{1} w \ldots + \binom{2n}{1} z = 0$, which are evidently independent of the arbitrary direction $(\alpha_1, \beta_1, \ldots \zeta_1)$,

that is to say, this ray of the principal plane lies in a fixed linear continuum of n dimensions. Since the same statement holds as to the conjugate ray and the conjugate ∞^n, we see at once, that the principal plane passing through some given radius vector, is determined by having a line in common with either of two fixed (and *central*) linear continua of n dimensions. Lest any doubt remain of the meaning of this proposition, let $(u, v, \dots z)$ be the given radius vector,

$$(u_1 \dots),\ (u_2 \dots), \dots (u_n \dots),$$

n independent rays in one fixed ∞^n,

$$(u'_1 \dots),\ (u'_2 \dots),\ \dots (u'_n \dots)$$

likewise n independent rays in the other fixed ∞^n, and form $2n$ equations such as

$$\alpha u + \alpha_1 u_1 + \alpha_2 u_2 \dots + \alpha_n u_n + \alpha_1' u_1' + \alpha_2' u_2' \dots + \alpha_n' u_n' = 0,$$

in respect to each axis of coordinates, and justly enough to determine the $2n$ ratios of the $2n+1$ indeterminates

$$\alpha,\ \alpha_1, \dots \alpha_n,\ \alpha_1', \dots \alpha_n'.$$

Then at varying the indeterminate θ, the ray

$$(\theta u + \alpha_1 u_1 + \alpha_2 u_2 \dots + \alpha_n u_n, \dots)$$

will describe the principal plane in question, as while the rays which such plane has in common with the two fixed continua ∞^n are represented by $\theta = 0$ and by $\theta = \alpha$.

We go on to transform the primitive displacement-system $\begin{pmatrix} 1, & 2, & 3, & \dots 2n \\ 1, & 2, & 3, & \dots 2n \end{pmatrix}$ into another, where any such two perpendicular directions $(\alpha_1, \beta_1, \dots)$, $(\alpha_2, \beta_2, \dots)$, as form a principal plane, are adopted as axes of coordinates, and complete them to the orthogonal transformation-matrix $\begin{pmatrix} \alpha, & \beta, & \gamma, & \dots \zeta \\ 1, & 2, & 3, & \dots 2n \end{pmatrix}$, so that $u = \Sigma_r \alpha_r t_r$, $v = \Sigma \beta_r t_r$, $\dots$, $z = \Sigma \zeta_r t_r$;

$$t_q' = \alpha_q u' + \beta_q v' + \dots + \zeta_q z' = \Sigma_r a_{qr} t_r.$$

Then

$$a_{qr} = \lambda (\alpha_q \alpha_r + \beta_q \beta_r \dots + \zeta_q \zeta_r)$$
$$+ \binom{1}{2} \binom{\alpha\beta}{qr} + \binom{1}{3} \binom{\alpha\gamma}{qr} \dots + \binom{2n-1}{2n} \binom{\epsilon\zeta}{qr},$$

whence $a_{rr} = \lambda$, and if $q \pm r$, then $a_{rq} = -a_{qr}$, since the coefficient of λ vanishes. In the expression of a_{1r} the co-

efficient of α_r is $\binom{2}{1}\beta_1+\binom{3}{1}\gamma_1\ldots+\binom{2n}{1}\zeta_1=-\mu\alpha_2$; that of β_r is $-\mu\beta_2$, &c., whence $a_{12}=-\mu$, $a_{13}=0$, $a_{14}=0$, ... $a_{1(2n)}=0$. Likewise, in the expression of a_{2r} the coefficient of α_r is $\binom{2}{1}\beta_2+\binom{3}{1}\gamma_2\ldots+\binom{2n}{1}\zeta_2=\mu\alpha_1$, &c.; accordingly also $a_{23}=0$, $a_{24}=0$, ... $a_{2(2n)}=0$. What we have gained by this transformation is that the binary link $\begin{vmatrix}\lambda, & -\mu\\ \mu, & \lambda\end{vmatrix}$ at the upper left-hand corner comes off, so that thenceforth we are only concerned with a displacement-substitution

$$(t_3'=\lambda t_3+a_{34}t_4+a_{35}t_5\ldots,\quad t_4'=a_{43}t_3+\lambda t_4\ldots,\ \&\text{c.})$$

of $2n-2$ dimensions, and may begin the same proceeding anew, when we will come down to $2n-4$ dimensions, and so on, until we arrive at a displacement-substitution, broken up into a set of n congruent binary links, such as $t'_{2r-1}=\lambda t_{2r-1}-\mu t_{2r}$, $t'_{2r}=\mu t_{2r-1}+\lambda t_{2r}$, where $r=1, 2, 3, \ldots n$.

In order to ascertain the number of independent elements of a skew and orthogonal matrix of order $2n$, we are to count first the angle whose cosine and sine are λ, μ as one element. The first principal plane then if fixed would count $4(n-1)$ elements; but since it has been found, by means of an arbitrary ray counting $2n-1$, which itself can move in such plane, that number is to be diminished by $(2n-1)-1$; and so the first principal plane counts only $2(n-1)$ elements, the second $2(n-2)$, and so on, the last but one counts two elements. Therefore, *a skew and orthogonal matrix of order* $2n$ *counts* n^2-n+1 *independent elements.* Consequently the skew property implies n^2-1 conditions, since it reduces the full number $2n^2-n$ of an orthogonal matrix whatever (but obviously of the same order) to n^2-n+1; that is to say, among the $2n^2+n-1$ conditions,

$$\binom{1}{1}=\binom{2}{2}=\binom{3}{3}\ldots=\binom{2n}{2n},\quad \binom{2}{1}=-\binom{1}{2},\ \&\text{c.}$$

there are only n^2-1 independent ones. It seems to me very difficult to establish this proposition in a direct way.

Bern, 27 *October*, 1865.

SOLUTION OF A PROBLEM OF ELIMINATION.

By Professor CAYLEY.

It is required to eliminate x, y from the equations

$$\left\| \begin{array}{ccccc} x^4, & x^3y, & x^2y^2, & xy^3, & y^4 \\ a, & b, & c, & d, & e \\ a', & b', & c', & d', & e' \\ a'', & b'', & c'', & d'', & e'' \end{array} \right\| = 0.$$

This system may be written

$$\begin{aligned} x^4 &= \Sigma\lambda a, \\ x^3y &= \Sigma\lambda b, \\ x^2y^2 &= \Sigma\lambda c, \\ xy^3 &= \Sigma\lambda d, \\ y^4 &= \Sigma\lambda e; \end{aligned}$$

if for shortness $\Sigma\lambda a = \lambda a + \lambda' a' + \lambda'' a''$, &c.

Or putting $$\frac{x}{y} = -k,$$

we have

$$\begin{aligned} \Sigma\lambda\,(a + kb) &= 0, \\ \Sigma\lambda\,(b + kc) &= 0, \\ \Sigma\lambda\,(c + kd) &= 0, \\ \Sigma\lambda\,(d + ke) &= 0; \end{aligned}$$

or, what is the same thing,

$$\begin{aligned} \lambda\,(a + kb) + \lambda'\,(a' + kb') + \lambda''\,(a'' + kb'') &= 0, \\ \lambda\,(b + kc) + \lambda'\,(b' + kc') + \lambda''\,(b'' + kc'') &= 0, \\ \lambda\,(c + kd) + \lambda'\,(c' + kd') + \lambda''\,(c'' + kd'') &= 0, \\ \lambda\,(d + ke) + \lambda'\,(d' + ke') + \lambda''\,(d'' + ke'') &= 0. \end{aligned}$$

And representing the columns

$$\begin{array}{lll} ab, & a'b', & a''b'', \\ bc, & b'c', & b''c'', \\ cd, & c'd', & c''d'', \\ de, & d'e', & d''e'', \end{array}$$

by $$1,\ 2,\ 3,\ 4,\ 5,\ 6,$$

each equation is of the type

$$\lambda\,(1+k2)+\lambda'\,(3+k4)+\lambda''\,(5+k6)=0.$$

Multiplying the several equations by the minors of 135, each with its proper sign, and adding, the terms independent of k disappear, the equation divides by k, and we find

$$\lambda 2135+\lambda' 4135+\lambda'' 6135=0.$$

Operating in a similar manner with the minors of 246, the terms in k disappear, and we find

$$\lambda 1246+\lambda' 3246+\lambda'' 5246=0.$$

Again, operating with the minors of $(146+236+245+k246)$, we find

$$\begin{aligned} &\lambda\ \{1236+1245+k\,(2146+1246)\}\\ &+\lambda'\ \{3146+3245+k\,(4236+3246)\}\\ &+\lambda''\ \{5146+5236+k\,(6245+5246)\}=0, \end{aligned}$$

where the terms in k disappear, and this is

$$\lambda\,(1236+1245)+\lambda'\,(3146+3245)+\lambda''\,(5146+5236)=0.$$

We have thus three linear equations, which written in a slightly different form are

$$\begin{array}{lllllll} \lambda 1235 & + & \lambda' 3451 & + & \lambda'' 5613 & =0, \\ \lambda\,(1236+1245) & + & \lambda'\,(3452+3461) & + & \lambda''\,(5614+5623) & =0, \\ \lambda 1246 & + & \lambda' 3462 & + & \lambda'' 5624 & =0, \end{array}$$

and thence eliminating $\lambda,\ \lambda',\ \lambda''$, we have

$$\begin{vmatrix} 1235, & 1236+1245, & 1246 \\ 3451, & 3452+3461, & 3462 \\ 5613, & 5614+5623, & 5624 \end{vmatrix}=0,$$

which is the required result. It may be remarked that the second and third column are obtained from the first by operating on it with Δ, $\frac{1}{2}\Delta^2$, if $\Delta=2\delta_1+4\delta_3+6\delta_5$. Or say the result is

$$(1,\ \Delta,\ \tfrac{1}{2}\Delta^2)\begin{vmatrix} 1235 \\ 3451 \\ 5613 \end{vmatrix}=0.$$

In like manner for the system

$$\left\|\begin{matrix} x^5, & x^4y, & x^3y^2, & x^2y^3, & xy^4, & y^5 \\ a, & b, & c, & d, & e, & f \\ a', & b', & c', & d', & e', & f' \\ a'', & b'', & c'', & d'', & e'', & f'' \\ a''', & b''', & c''', & d''', & e''', & f''' \end{matrix}\right\| = 0.$$

If the columns are

$$\begin{matrix} ab, & a'b', & a''b'', & a'''b''', \\ bc, & b'c', & b''c'', & b'''c''', \\ cd, & c'd', & c''d'', & c'''d''', \\ de, & d'e', & d''e'', & d'''e''', \\ ef, & e'f', & e''f'', & e'''f''', \end{matrix}$$

$$= 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8;$$

then the result is

$$(1,\ \Delta,\ \tfrac{1}{2}\Delta^2,\ \tfrac{1}{6}\Delta^3) \left|\begin{matrix} 12357 \\ 34571 \\ 56713 \\ 78135 \end{matrix}\right| = 0,$$

where $$\Delta = 2\delta_1 + 4\delta_3 + 6\delta_5 + 8\delta_7.$$

ON THE GENERALIZATION OF CERTAIN FORMULÆ INVESTIGATED BY MR. BLISSARD.

By Worontzof, M.M.

I. On the Sums of the Reciprocals, of their Products and Powers.

1. Let $$\frac{1}{1.2.3\ldots m} + \frac{1}{2.3.4\ldots(m+1)} + \ldots$$
$$+ \frac{1}{x(x+1)(x+2)\ldots(x+m-1)} = \Sigma_x^{(m)},$$

then from the equivalence

$$\left\{1-n(1+\Delta)+\frac{n(n-1)}{1.2}(1+\Delta)^2-\ldots+(-1)^n(1+\Delta)^n\right\}\Sigma_x^{(m)}$$
$$=(-1)^n\Delta^n\Sigma_x^{(m)}\ (\Delta x=1),$$

we have $\Sigma_x^{(m)}-n\Sigma_{x+1}^{(m)}+\frac{n(n-1)}{1.2}\Sigma_{x+2}^{(m)}-\ldots+(-1)^n\Sigma_{x+n}^{(m)}$

$$=-\frac{\Gamma(m+n-1)\,\Gamma(x+1)}{\Gamma m\,\Gamma(x+m+n)}\ \ldots\ldots\ldots(1),$$

and making $m=1$,

$$\Sigma_x-n\Sigma_{x+1}+\frac{n(n-1)}{1.2}\Sigma_{x+2}-\ldots+(-1)^n\Sigma_{x+n}$$
$$=-\frac{\Gamma n\,\Gamma(x+1)}{\Gamma(x+n+1)}\ \ldots\ldots\ldots(2).*$$

Cor. Let $n=1$ and $x=0$, then $\Sigma_0^{(m)}=0$ and $\Sigma_0=0$; therefore

$$n\Sigma_1^{(m)}-\frac{n(n-1)}{1.2}\Sigma_2^{(m)}+\ldots+(-1)^{n-1}\Sigma_n^{(m)}=\frac{1}{(m+n-1)\,\Gamma m},$$

and $$n\Sigma_1-\frac{n(n-1)}{1.2}\Sigma_2+\ldots+(-1)^{n-1}\Sigma_n=\frac{1}{n}.$$

2. From the equivalence

$$(1+\Delta)^n\Sigma_x^m=\left\{1+n\Delta+\frac{n(n-1)}{1.2}\Delta^2+\ldots+\Delta^n\right\}\Sigma_x^m\ (\Delta x=1),$$

we obtain the following formulæ:

$$\Sigma_{x+n}^{(m)}=\Sigma_x^{(m)}+\frac{\Gamma(x+1)}{\Gamma(x+m)}\left\{\frac{n}{x+m}-\frac{n(n-1)}{1.2}\,\frac{m}{(x+m)(x+m+1)}\right.$$
$$+\frac{n(n-1)(n-2)}{1.2.3}\,\frac{m(m+1)}{(x+m)(x+m+1)(x+m+2)}-\ldots$$
$$\left.+(-1)^{n-1}\frac{m(m+1)\ldots(m+n-2)}{(x+m)(x+m+1)\ldots(x+m+n-1)}\right\}\ \ldots\ldots\ldots(3),$$

$$\Sigma_n^{(m)}=\frac{1}{\Gamma m}\left\{\frac{n}{m}-\frac{n(n-1)}{1.2}\cdot\frac{1}{m+1}+\frac{n(n-1)(n-2)}{1.2.3}\cdot\frac{1}{m+2}-\ldots\right.$$
$$\left.+(-1)^{n-1}\frac{1}{m+n-1}\right\}\ldots\ldots\ldots\ldots(4),$$

* Blissard, "On the Sums of Reciprocals," *Quarterly Mathematical Journal*, Vol. VI., p. 255, Art. 11.

$$\Sigma_{x+n}=\Sigma_x+\frac{n}{x+1}-\frac{n(n-1)}{(x+1)(x+2)}\cdot\frac{1}{2}+\frac{n(n-1)(n-2)}{(x+1)(x+2)(x+3)}\cdot\frac{1}{3}-\ldots$$

$$+(-1)^{n-1}\frac{n(n-1)\ldots 2.1}{(x+1)(x+2)\ldots(x+n)}\cdot\frac{1}{n}\,\ldots\ldots\ldots(5)^*,$$

and $\Sigma_n=\frac{n}{1}-\frac{n(n-1)}{1.2}\cdot\frac{1}{2}+\frac{n(n-1)(n-2)}{1.2.3}\cdot\frac{1}{3}-\ldots+(-1)^{n-1}\frac{1}{n}$

$$\ldots\ldots\ldots\ldots\ldots\ldots(6).$$

3. The formula

$$\{1+(1+\Delta)+(1+\Delta)^2+\ldots+(1+\Delta)^{n-1}\}\,\Delta^m\Sigma_x$$

$$=(-1)^{m-1}\left\{1-(m-1)(1+\Delta)+\frac{(m-1)(m-2)}{1.2}(1+\Delta^2)+\ldots\right.$$

$$\left.+(-1)^{m-1}(1+\Delta)^{m-1}\right\}[(1+\Delta)^n-1]\,\Sigma_x\,(\Delta x=1)$$

gives $\Sigma_{x+n}{}^{(m)}-\Sigma_x{}^{(m)}=\frac{1}{\Gamma m}\left\{[\Sigma_{x+n}-\Sigma_x]-(m-1)[\Sigma_{x+n+1}-\Sigma_{x+1}]\right.$

$$\left.+\frac{(m-1)(m-2)}{1.2}[\Sigma_{x+n+2}-\Sigma_{x+2}]-\ldots+(-1)^{m-1}[\Sigma_{x+n+m-1}-\Sigma_{x+m-1}]\right\}$$

$$\ldots\ldots\ldots\ldots\ldots(7).$$

Hence, supposing $x=0$,

$$\Sigma_n{}^{(m)}=\frac{1}{\Gamma m}\{\Sigma_n-(m-1)[\Sigma_{n+1}-\Sigma_1]+\frac{(m-1)(m-2)}{1.2}[\Sigma_{n+2}-\Sigma_2]$$

$$-\ldots+(-1)^{m-1}[\Sigma_{n+m-1}-\Sigma_{m-1}]\}\,\ldots\ldots\ldots\ldots(8).\dagger$$

4. By virtue of the equivalence

$$\left\{(1+\Delta)^{m+n}-(m+n)(1+\Delta)^{m+n-1}+\frac{(mn+n)(m+n-1)}{1.2}(1+\Delta)^{m+n-2}-\ldots\right.$$

$$\left.+(-1)^{m+n}\right\}\Delta(1+\Delta)^{-(m+n+1)}\Sigma_x$$

$$=\left\{(1+\Delta)^{n-1}-(n-1)(1+\Delta)^{n-2}+\frac{(n-1)(n-2)}{1.2}(1+\Delta)^{n-3}-\ldots\right.$$

$$\left.+(-1)^{n-1}\right\}\Delta^{m+2}(1+\Delta)^{-(m+n+1)}\,\Sigma_x\,(\Delta x=1),$$

* Blissard, "On the Sums of Reciprocals," Art. 6, p. 250.

† It is known that $\Sigma_n{}^{(m)}$ can also be expressed as follows:

$$\Sigma_n{}^{(m)}=\frac{1}{(m-1)\,\Gamma m}\left\{1-\frac{\Gamma m\,\Gamma(n+1)}{\Gamma(m+n)}\right\},$$

hence $\Sigma_n{}^{(n)}=\frac{1}{(n-1)\,\Gamma n}\left\{1-\frac{\sqrt{(\pi)}\,\Gamma(n+1)}{2^{2n-1}\,\Gamma(n+\frac{1}{2})}\right\}$ and $\Sigma_\infty{}^{(m)}=\frac{1}{(m-1)\,\Gamma m}$.

we have

$$\frac{1}{x}-(m+n)\frac{1}{x-1}+\frac{(m+n)(m+n-1)}{1.2}\frac{1}{x-2}-\ldots$$

$$+(-1)^{n-1}\frac{(m+n)(m+n-1)\ldots(m+2)}{1.2.3\ldots(n-1)}\frac{1}{x-n+1}$$

$$+(-1)^{n}\frac{(m+n)(m+n-1)\ldots(m+1)}{1.2.3\ldots n}\frac{1}{x-n}$$

$$+(-1)^{n+1}\frac{(m+n)(m+n-1)\ldots m}{1.2.3\ldots(n+1)}\frac{1}{x-n-1}$$

$$+(-1)^{n+2}\frac{(m+n)(m+n-1)\ldots(m-1)}{1.2.3\ldots(n+2)}\frac{1}{x-n-2}+\ldots+(-1)^{m+n}\frac{1}{x-n-m}$$

$$=1.2.3\ldots(m+1)(-1)^{m+1}\left\{\frac{1}{x(x-1)\ldots(x-m-1)}\right.$$

$$-(n-1)\frac{1}{(x-1)(x-2)\ldots(x-m-2)}+\frac{(n-1)(n-2)}{1.2}\frac{1}{(x-2)(x-3)\ldots(x-m-3)}$$

$$+\ldots+(-1)^{n-m-2}\frac{1}{(x-n+m+2)\ldots(x-n+1)}\frac{(n-1)(n-2)\ldots(n-m-1)}{1.2.3\ldots(m+1)}$$

$$+(-1)^{n-m-1}\frac{(n-1)(n-2)\ldots(n-m)}{1.2.3\ldots m}\frac{1}{(x-n+m+1)\ldots(x-n)}$$

$$+(-1)^{n-m}\frac{(n-1)(n-2)\ldots(n-m+1)}{1.2.3\ldots(m-1)}\frac{1}{(x-n+m)\ldots(x-n-1)}$$

$$\left.+\ldots+(-1)^{n-1}\frac{1}{(x-n+1)(x-n)\ldots(x-n-m)}\right\}.$$

Let $x=n$, then since

$$\frac{1}{n}-(m+n)\frac{1}{n-1}+\ldots+(-1)^{n-1}\frac{(m+n)(m+n-1)\ldots(m+2)}{1.2.3\ldots(n-1)}$$

$$=\frac{(m+1)(m+2)\ldots(m+n)}{1.2.3\ldots n}(-1)^{n-1}(\Sigma_{m+n}-\Sigma_m)$$

$$=\frac{(n+1)(n+2)\ldots(n+m)}{1.2.3\ldots m}(-1)^{n-1}(\Sigma_{m+n}-\Sigma_m),$$

$$(-1)^{n+2}\left\{\frac{(m+n)(m+n-1)\ldots m}{1.2.3\ldots(n+1)}-\frac{(m+n)(m+n-1)\ldots(m-1)}{1.2.3\ldots(n+2)}\cdot\frac{1}{2}+\ldots\right.$$

$$\left.+(-1)^{m-1}\frac{1}{m}\right\}=(-1)^{n-1}\frac{(n+1)(n+2)\ldots(n+m)}{1.2.3\ldots m}(\Sigma_n-\Sigma_{m+n}),$$

and

$$\left[(-1)^{m+1}\frac{1.2.3\ldots(m+1)}{x-n}\left\{(-1)^{n-m-1}\frac{(n-1)(n-2)\ldots(n-m)}{1.2.3\ldots m}\right.\right.$$

$$\times\frac{1}{(x-n+m+1)\ldots(x-n+1)}$$

$$+(-1)^{n-m}\frac{(n-1)(n-2)\ldots(n-m+1)}{1.2.3\ldots(m-1)}\;\frac{1}{(x-n+m)\ldots(x-n+1)(x-n-1)}$$

$$+\ldots+(-1)^{n-1}\frac{1}{(x-n+1)(x-n-1)\ldots(x-n-m)}$$

$$\left.\left.-(-1)^{n+m+1}\frac{(m+n)(m+n-1)\ldots(m+2)}{1.2.3\ldots n,\ 1.2.3\ldots m}\right\}\right]_{x=n}=\frac{0}{0}$$

$$=(-1)^{n-1}\left\{\frac{(n-1)(n-2)\ldots(n-m)}{1.2.3\ldots m}\Sigma_{m+1}\right.$$

$$+\frac{(m+1)}{1}\;\frac{(n-1)(n-2)\ldots(n-m+1)}{1.2.3\ldots(m-1)}(\Sigma_m-\Sigma_1)$$

$$+\frac{(m+1)m}{1.2}\;\frac{(n-1)(n-2)\ldots(n-m+2)}{1.2.3\ldots(m-2)}(\Sigma_{m-1}-\Sigma_2)+\ldots$$

$$\left.+\frac{(m+1)m}{1.2}\;\frac{n-1}{1}(\Sigma_2-\Sigma_{m-1})+\frac{(m+1)}{1}(\Sigma_1-\Sigma_m)\right\},$$

we obtain Mr. Blissard's general formulæ,*

$$\Sigma_n=\Sigma_m+\frac{1.2.3\ldots m}{(n+1)(n+2)\ldots(n+m)}\left[\frac{(n-1)(n-2)\ldots(n-m)}{1.2.3\ldots m}\Sigma_{m+1}\right.$$

$$+\frac{m+1}{1}\;\frac{(n-1)(n-2)\ldots(n-m+1)}{1.2.3\ldots(m-1)}(\Sigma_m-\Sigma_1)$$

$$+\frac{(m+1)m}{1.2}\;\frac{(n-1)(n-2)\ldots.(n-m+2)}{1.2.3\ldots(m-2)}(\Sigma_{m-1}-\Sigma_2)+\ldots$$

$$+\frac{(m+1)m}{1.2}\;\frac{n-1}{1}(\Sigma_2-\Sigma_{m-1})+\frac{m+1}{1}(\Sigma_1-\Sigma_m)$$

$$+1.2.3\ldots(m+1)\frac{(-1)^{m+n}}{n}\left\{\frac{1}{(n-1)(n-2)\ldots(n-m-1)}\right.$$

$$-n\frac{1}{(n-2)(n-3)\ldots(n-m-2)}+\frac{n(n-1)}{1.2}\;\frac{1}{(n-3)(n-4)\ldots(n-m-3)}-\ldots$$

$$\left.\left.+(-1)^{n-m-2}\frac{n(n-1)\ldots(n-m-1)}{1.2.3\ldots(m+2)}\cdot\frac{1}{1.2.3\ldots(m+1)}\right\}\right]\ldots\ldots(9),$$

* Blissard, "On the Sums of Reciprocals," Arts. 1 and 2, pp. 242-43.

and

$$\Sigma_n = \Sigma_m + \frac{1.2.3\ldots m}{(n+1)(n+2)\ldots(n+m)}\left[\frac{(n-1)(n-2)\ldots(n-m)}{1.2.3\ldots m}\Sigma_{m+1}\right.$$

$$+\frac{m+1}{1}\,\frac{(n-1)(n-2)\ldots(n-m+1)}{1.2.3\ldots(m-1)}(\Sigma_m - \Sigma_1)$$

$$+\frac{(m+1)m}{1.2}\,\frac{(n-1)(n-2)\ldots(n-m+2)}{1.2.3\ldots(m-2)}(\Sigma_{m-1} - \Sigma_2)+\ldots$$

$$+\frac{(m+1)m}{1.2}\,\frac{n-1}{1}(\Sigma_2 - \Sigma_{m-1}) + \frac{m+1}{1}(\Sigma_1 - \Sigma_m)$$

$$+1.2.3\ldots(m+1)\left\{\frac{(n-1)(n-2)\ldots(n-m-1)}{1.2.3\ldots(m+1)}\,\frac{1}{1.2.3\ldots(m+2)}\right.$$

$$-\frac{(n-1)(n-2)\ldots(n-m-2)}{1.2.3\ldots(m+2)}\cdot\frac{1}{2.3.4\ldots(m+3)}+\ldots$$

$$\left.\left.+(-1)^{n-m-2}\frac{1}{n(n-1)(n-2)\ldots(n-m-1)}\right\}\right] \ldots\ldots\ldots\ldots\ (10).$$

Cor. If $m = n-1$, then

$$\frac{(n+1)(n+2)\ldots(2n-1)}{1.2.3\ldots n} = \Sigma_n + n(n-1)(\Sigma_{n-1} - \Sigma_1)$$

$$+\frac{n(n-1)}{1.2}\,\frac{(n-1)(n-2)}{1.2}(\Sigma_{n-2} - \Sigma_2)+\ldots+n(\Sigma_1 - \Sigma_{n-1}).$$

5. From the equivalence

$$\Delta^{n+1}\Sigma_x^{m+2} = (-1)^n\left\{1 - n(1+\Delta) + \frac{n(n-1)}{1.2}(1+\Delta)^2 - \ldots\right.$$

$$\left.+(-1)^n(1+\Delta)^n\right\}\Delta\Sigma_x^{m+2}\ (\Delta x = 1),$$

we have $\dfrac{(m+2)(m+3)\ldots(m+n+1)}{(x+1)(x+2)\ldots(x+m+n+2)} = \dfrac{1}{(x+1)(x+2)\ldots(x+m+2)}$

$$-n\frac{1}{(x+2)(x+3)\ldots(x+m+3)} + \frac{n(n-1)}{1.2}\,\frac{1}{(x+3)(x+4)\ldots(x+m+4)} - \ldots$$

$$+(-1)^n\frac{1}{(x+n+1)(x+n+2)\ldots(x+n+m+2)}.$$

Let $x=-2$, then since

$$\left[\frac{1}{x+2}\left\{\frac{(m+2)(m+3)\ldots(m+n+1)}{(x+1)(x+3)\ldots(x+m+n+2)}-\frac{1}{(x+1)(x+3)\ldots(x+m+2)}\right.\right.$$
$$\left.\left.+n\frac{1}{(x+3)(x+4)\ldots(x+m+3)}\right\}\right]_{x=-2}=\frac{0}{0}$$
$$=\frac{m+n+1}{1.2.3\ldots(m+1)}\left\{\Sigma_{m+n}-\Sigma_m-\frac{n(m+2)}{(m+n+1)(m+1)}\right\},$$

we obtain

$$\Sigma_{m+n}-\Sigma_m=\frac{1}{m+n+1}\left\{\frac{n(m+2)}{m+1}+1.2.3\ldots(m+1)\left[\frac{n(n-1)}{1.2}\frac{1}{1.2.3\ldots(m+2)}\right.\right.$$
$$-\frac{n(n-1)(n-2)}{1.2.3}\frac{1}{2.3.4\ldots(m+3)}+\ldots$$
$$\left.\left.+(-1)^{n-2}\frac{1}{(n-1)(n)(n+1)\ldots(n+m)}\right]\right\}\ldots\ldots\ldots(11).$$

Hence if $m=0$,

$$\Sigma_n=\frac{1}{n+1}\left\{2n+\frac{n(n-1)}{1.2}\frac{1}{1.2}-\frac{n(n-1)(n-2)}{1.2.3}\frac{1}{2.3}+\ldots\right.$$
$$\left.+(-1)^{n-2}\frac{1}{(n-1)n}\right\}\ldots\ldots\ldots(12)^*.$$

6. From the formula

$$\Delta^{n-1}\frac{\Sigma_x}{x(x+1)}=(-1)^{n-1}\left\{1-(n-1)(1+\Delta)+\frac{(n-1)(n-2)}{1.2}(1+\Delta)^2-\ldots\right.$$
$$\left.+(-1)^{n-1}(1+\Delta)^{n-1}\right\}\frac{\Sigma_x}{x(x+1)}\ (\Delta x=1),$$

we have $\Sigma_{x+n-1}-\Sigma_x=n\left\{\frac{1}{x+1}+\frac{1}{2(x+2)}+\frac{1}{3(x+3)}+\ldots\right.$

$$\left.+\frac{1}{(n-1)(x+n-1)}-\frac{\Sigma_x}{x}\right\}+\frac{\Gamma(x+n+1)}{\Gamma n\,\Gamma(x+1)}\left\{\frac{\Sigma_x}{x(x+1)}-(n-1)\frac{\Sigma_{x+1}}{(x+1)(x+2)}\right.$$
$$\left.+\frac{(n-1)(n-2)}{1.2}\frac{\Sigma_{x+2}}{(x+2)(x+3)}-\ldots+(-1)^{n-1}\frac{\Sigma_{x+n-1}}{(x+n-1)(x+n)}\right\}\ldots(13).$$

Hence if $x=n$,

$$\Sigma_{2n-1}=n\left\{\frac{1}{n+1}+\frac{1}{2(n+2)}+\frac{1}{3(n+3)}+\ldots+\frac{1}{(n-1)(2n-1)}\right\}$$
$$+\frac{\Gamma(2n+1)}{\Gamma n\,\Gamma(n+1)}\left\{\frac{\Sigma_n}{n(n+1)}-(n-1)\frac{\Sigma_{n+1}}{(n+1)(n+2)}\right.$$
$$\left.+\frac{(n-1)(n-2)}{1.2}\frac{\Sigma_{n+2}}{(n+2)(n+3)}-\ldots+(-1)^{n-1}\frac{\Sigma_{2n-1}}{(2n-1)2n}\right\}\ldots(14),$$

* Blissard, "On the Sums of Reciprocals," Art. 6, p. 249.

if $x=1$,

$$\Sigma_n=\frac{(n+1)\,n}{1.2}\,\Sigma_1-\frac{(n+1)\,n\,(n-1)}{1.2.3}\,\Sigma_2+\ldots+(-1)^{n-1}\,\Sigma_n\ldots(15),^*$$

and if $x=0$, changing n into $n+1$,

$$\Sigma_n=(n+1)\;\Sigma_n(2)$$
$$-\left\{\frac{(n+1)\,n}{1.2}\,\frac{\Sigma_1}{1}-\frac{(n+1)\,n\,(n-1)}{1.2.3}\,\frac{\Sigma_2}{2}+\ldots+(-1)^{n-1}\,\frac{\Sigma_n}{n}\right\}\ldots(16),$$

where $$\Sigma_n(2)=\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\ldots+\frac{1}{n^2}.$$

7. Comparing with (16) the formula

$$\Sigma_1(2)+\Sigma_2(2)+\ldots+\Sigma_n(2)$$
$$=\frac{(n+1)\,n}{1.2}\,\frac{\Sigma_1}{1}-\frac{(n+1)\,n\,(n-1)}{1.2.3}\,\frac{\Sigma_2}{2}+\ldots+(-1)^{n-1}\,\frac{\Sigma_n}{n}\ldots(17),$$

(see Art. 10, form. (28)), we have

$$\Sigma_n=n\,\Sigma_n(2)-\{\Sigma_1(2)+\Sigma_2(2)+\ldots+\Sigma_{n-1}(2)\}\ldots(18),$$

and also $$\Sigma_{m+n}-\Sigma_m=(m+n)\,\Sigma_{m+n}(2)-m\,\Sigma_m(2)$$
$$-\{\Sigma_m(2)+\Sigma_{m+1}(2)+\ldots+\Sigma_{m+n-1}(2)\}\ldots\ldots(19).$$

8. Now let

$$\phi_0\;(x,n)=1,$$

$$\phi_1\;(x,n)=\frac{1}{x+1}\,\phi_0\;(x,1)+\frac{1}{x+2}\,\phi_0\;(x,2)+\ldots+\frac{1}{x+n}\,\phi_0\;(x,n)$$
$$=\Sigma_{x+n}-\Sigma_x,$$

$$\phi_2\;(x,n)=\frac{1}{x+1}\,\phi_1\;(x,1)+\frac{1}{x+2}\,\phi_1\;(x,2)+\ldots+\frac{1}{x+n}\,\phi_1\;(x,n),$$

$$\phi_3\;(x,n)=\frac{1}{x+1}\,\phi_2\;(x,1)+\frac{1}{(x+2)}\,\phi_2\;(x,2)+\ldots+\frac{1}{x+n}\,\phi_2\;(x,n),$$

....

$$\phi_{\nu-1}(x,n)=\frac{1}{x+1}\,\phi_{\nu-2}(x,1)+\frac{1}{x+2}\,\phi_{\nu-2}(x,2)+\ldots+\frac{1}{x+n}\,\phi_{\nu-2}(x,n),$$

(Blissard, "Researches in Analysis," *Quarterly Journal*, 1865, No. 27, p. 226).

* Blissard, "On the Sums of Reciprocals." Art. 12, p. 255, (1).

No. 31. February, 1867.

THE QUARTERLY JOURNAL OF PURE AND APPLIED MATHEMATICS.

EDITED BY

J. J. SYLVESTER, M.A., F.R.S.,

PROFESSOR OF MATHEMATICS IN THE ROYAL MILITARY ACADEMY, WOOLWICH; AND

N. M. FERRERS, M.A.,

FELLOW AND TUTOR OF GONVILLE AND CAIUS COLLEGE, CAMBRIDGE:

ASSISTED BY

G. G. STOKES, M.A., F.R.S.,

LUCASIAN PROFESSOR OF MATHEMATICS IN THE UNIVERSITY OF CAMBRIDGE;

A. CAYLEY, M.A., F.R.S.,

SADLERIAN PROFESSOR OF MATHEMATICS IN THE UNIVERSITY OF CAMBRIDGE; AND

M. HERMITE,

CORRESPONDING EDITOR IN PARIS.

ὅ τι οὐσία πρὸς γένεσιν, ἐπιστημὴ πρὸς πίστιν καὶ διάνοια πρὸς εἰκασίαν ἐστι.

LONDON:

LONGMANS, GREEN, AND CO.,

PATERNOSTER ROW.

1867.

W. METCALFE, PRINTER, GREEN STREET, CAMBRIDGE.

PRICE FIVE SHILLINGS.

CONTENTS.

NOTICES TO CORRESPONDENTS.

The following Papers have been received:

Mr. Walton, "On the Symbol of Operation $x\dfrac{d}{dx}$;" "On the Debility of Large Trees and Animals;" "On Biangular Coordinates;" and "Note on the Lunar Theory."

Professor Cayley, "Specimen Table $M \equiv a^{\alpha} b^{\beta}$ (mod. N) for any Prime or Composite Modulus;" "Tables of the Binary Cubic Forms for the Negative Determinants, $\equiv 0$ (Mod. 4), from -4 to -400; and $\equiv 1$ (Mod. 4), from -3 to -99; and for five Irregular Negative Determinants;" and "On a certain Envelope depending on a Triangle Inscribed in a Circle."

Mr. J. Blissard, "On the Properties of the $\Delta^m 0^n$ Class of Numbers."

Mr. F. Wolff, "Critical Examination of Euclid's First Principles compared to those of Modern Geometry, Ancient and Modern Analysis."

Mr. H. M. Jeffery, "On Plane Conics."

Notice to Subscribers.—The first Seven Volumes of the 'Journal,' bound in calf, gilt, or any of the back Numbers, can be had of the Publishers, or at the Printing Office, Green Street, Cambridge.

No. 32 will be published in May, 1867.

Then since

$$\Delta^{n-1}\frac{1}{(x+1)^v}=(-1)^{n-1}\left\{1-(n-1)(1+\Delta)+\frac{(n-1)(n-2)}{1.2}(1+\Delta)^2-\dots\right.$$
$$\left.+(-1)^{n-1}(1+\Delta)^{n-1}\right\}\frac{1}{(x+1)^v}\,(\Delta x=1)$$
$$=(-1)^n\left\{1-n(1+\Delta)+\frac{n(n-1)}{1.2}(1+\Delta)^2-\dots\right.$$
$$\left.+(-1)^n(1+\Delta)^n\right\}\Sigma_x(v)$$
$$=(-1)^{n+1}\left\{1-(n+1)(1+\Delta)+\frac{(n+1)n}{1.2}(1+\Delta)^2-\dots\right.$$
$$\left.+(-1)^{n+1}(1+\Delta)^{n+1}\right\}\{\Sigma_1(v)+\Sigma_2(v)+\Sigma_3(v)+\dots+\Sigma_{x-1}(v)\},$$

where $\Sigma_x(v)=\frac{1}{1^v}+\frac{1}{2^v}+\frac{1}{3^v}+\dots+\frac{1}{x^v}$,

we have

$$\phi_{v-1}(x,n)=\frac{\Gamma(x+n+1)}{\Gamma n\,\Gamma x+1}\left\{\frac{1}{(x+1)^v}-(n-1)\frac{1}{(x+2)^v}\right.$$
$$\left.+\frac{(n-1)(n-2)}{1.2}\frac{1}{(x+3)^v}-\dots+(-1)^{n-1}\frac{1}{(x+n)^v}\right\}\dots(20)$$
$$=-\frac{\Gamma(x+n+1)}{\Gamma n\,\Gamma(x+1)}\left\{\Sigma_x(v)-n\,\Sigma_{x+1}(v)\right.$$
$$\left.+\frac{n(n-1)}{1.2}\Sigma_{x+2}(v)-\dots+(-1)^n\Sigma_{x+n}(v)\right\}\dots(21)$$
$$=-\frac{\Gamma(x+n+1)}{\Gamma n\,\Gamma(x+1)}\left\{(n+1)\,\Sigma_x(v)-\frac{(n+1)n}{1.2}[\Sigma_x(v)+\Sigma_{x+1}(v)]\right.$$
$$+\frac{(n+1)\,n(n-1)}{1.2.3}[\Sigma_x(v)+\Sigma_{x+1}(v)+\Sigma_{x+2}(v)]-\dots$$
$$\left.+(-1)^n[\Sigma_x(v)+\dots+\Sigma_{x+n}(v)]\right\}\dots\dots\dots\dots(22).$$

Hence $(v=1)$... formula (2) of Art. 1. Also

$$\frac{\Gamma n\,\Gamma x+1}{\Gamma(x+n+1)}=\frac{1}{x+1}-(n-1)\frac{1}{x+2}$$
$$+\frac{(n-1)(n-2)}{1.2}\frac{1}{x+3}-\dots+(-1)^{n-1}\frac{1}{x+n}$$
$$=-\left\{(n+1)\Sigma_x-\frac{(n+1)n}{1.2}(\Sigma_x+\Sigma_{x+1})+\dots+(-1)^n(\Sigma_x+\Sigma_{x+1}+\dots+\Sigma_{x+n})\right\},$$

$$(x=0),\ \frac{1}{n}=1-(n-1)\frac{1}{2}+\frac{(n-1)(n-2)}{1.2}\frac{1}{3}-\ldots+(-1)^{n-1}\frac{1}{n},$$

$(v=2)$,

$$\Sigma_{x+n}-\Sigma_x=\frac{\Gamma(x+n+1)}{\Gamma n\ \Gamma(x+1)}\left\{\frac{1}{(x+1)^2}-(n-1)\frac{1}{(x+2)^2}\right.$$
$$\left.+\frac{(n-1)(n-2)}{1.2}\frac{1}{(x+3)^2}-\ldots+(-1)^{n-1}\frac{1}{(x+n)^2}\right\}*$$
$$=-\frac{\Gamma x+n+1}{\Gamma n\ \Gamma(x+1)}\left\{\Sigma_x(2)-n\,\Sigma_{x+1}(2)\right.$$
$$\left.+\frac{n(n-1)}{1.2}\Sigma_{x+2}(2)-\ldots+(-1)^n\Sigma_{x+n}(2)\right\}$$
$$=-\frac{\Gamma(x+n+1)}{\Gamma n\ \Gamma(x+1)}\left\{(n+1)\,\Sigma_x(2)\right.$$
$$\left.-\frac{(n+1)n}{1.2}[\Sigma_x(2)+\Sigma_{x+1}(2)]+\ldots\right\}\ldots(23),$$

$(x=0)$,

$$\frac{\Sigma_n}{n}=1-(n-1)\frac{1}{2^2}+\frac{(n-1)(n-2)}{1.2}\frac{1}{3^2}-\ldots+(-1)^{n-1}\frac{1}{n^2}$$
$$=n\,\Sigma_1(2)-\frac{n(n-1)}{1.2}\Sigma_2(2)+\ldots+(-1)^{n-1}\Sigma_n(2)$$
$$=\frac{(n+1)n}{1.2}\Sigma_1(2)-\frac{(n+1)n(n-1)}{1.2.3}[\Sigma_1(2)+\Sigma_2(2)]+\ldots$$
$$+(-1)^{n-1}[\Sigma_1(2)+\Sigma_2(2)+\ldots+\Sigma_n(2)]$$
$$=\frac{(n+1)n}{1.2}[2\,\Sigma_2(2)-\Sigma_2]-\frac{(n+1)n(n-1)}{1.2.3}[3\,\Sigma_3(2)-\Sigma_3]+\ldots$$
$$+(-1)^{n-1}[(n+1)\ \Sigma(2)-\Sigma_{n+1}],$$

$$(v=3),\ \frac{\Sigma_{x+1}-\Sigma_x}{x+1}+\frac{\Sigma_{x+2}-\Sigma_x}{x+2}+\ldots+\frac{\Sigma_{x+n}-\Sigma_x}{x+n}$$
$$=\frac{\Sigma_{x+1}}{x+1}+\frac{\Sigma_{x+2}}{x+2}+\ldots+\frac{\Sigma_{x+n}}{x+n}-\Sigma_x(\Sigma_{x+n}-\Sigma_x)$$
$$=\frac{\Gamma(x+n+1)}{\Gamma n\ \Gamma(x+1)}\left\{\frac{1}{(x+1)^3}-(n-1)\frac{1}{(x+2)^3}+\frac{(n-1)(n-2)}{1.2}\frac{1}{(x+3)^3}-\ldots\right.$$

* Blissard, "On the Sums of Reciprocals," Art. 6, p. 249, Ex. 3.

$$+(-1)^{n-1}\frac{1}{(x+n)^3}\Big\} = -\frac{\Gamma(x+n+1)}{\Gamma n\,\Gamma(x+1)}\Big\{\Sigma_x(3) - n\,\Sigma_{x+1}(3)$$

$$+\frac{n(n-1)}{1.2}\Sigma_{x+2}(3) - \ldots + (-1)^n\,\Sigma_{x+n}(3)\Big\} \ldots\ldots\ldots\ldots\ldots\ldots(24);$$

$$(x=0),\ \frac{\Sigma_1}{1}+\frac{\Sigma_2}{2}+\frac{\Sigma_3}{3}+\ldots+\frac{\Sigma_n}{n}$$

$$= n\left\{\frac{1}{1^3}-(n-1)\frac{1}{2^3}+\frac{(n-1)(n-2)}{1.2}\frac{1}{3^3}-\ldots+(-1)^{n-1}\frac{1}{n^3}\right\}$$

$$= n\left\{n\,\Sigma_1(3) - \frac{n(n-1)}{1.2}\Sigma_2(3) + \ldots + (-1)^{n-1}\,\Sigma_n(3)\right\}$$

$$= \tfrac{1}{2}\{\Sigma_n(2) + (\Sigma_n)^2\}.$$

9. From the equivalence

$$(1+\Delta)^{n-1}\frac{1}{(x+1)^v} = \Big\{1+(n-1)\Delta + \frac{(n-1)(n-2)}{1.2}\Delta^2 + \ldots$$

$$+\Delta^{n-1}\Big\}\frac{1}{(x+1)^v}\ (\Delta x = 1),$$

we have
$$\frac{1}{(x+n)^v} = \frac{1}{x+1}\phi_{v-1}(x,1) - \frac{n-1}{(x+1)(x+2)}\phi_{v-1}(x,2)$$

$$+\frac{(n-1)(n-2)}{(x+1)(x+2)(x+3)}\phi_{v-1}(x,3) - \ldots$$

$$+(-1)^{n-1}\frac{(n-1)(n-2)\ldots 3.2.1}{(x+1)(x+2)\ldots(x+n)}\phi_{v-1}(x,n),$$

(Blissard, "Researches in Analysis," *Quarterly Journal*, No. XXVII., p. 226), since

$$\Delta^p\frac{1}{(x+1)^v} = \frac{\Gamma(p+1)\,\Gamma(x+1)}{\Gamma(x+p+2)}(-1)^p\,\phi_{v-1}(x, p+1).$$

10. From the formulæ

$$\{1+(1+\Delta)+(1+\Delta)^2+\ldots+(1+\Delta)^{n-1}\}\frac{1}{(x+1)^v}$$

$$= \left\{n + \frac{n(n-1)}{1.2}\Delta + \frac{n(n-1)(n-2)}{1.2.3}\Delta^2 + \ldots + \Delta^{n-1}\right\}\frac{1}{(x+1)^v},$$

and $\{1+(1+\Delta)+(1+\Delta)^2+\ldots+(1+\Delta)^{n-1}\}\,\Sigma_x(v)$

$$=\left\{n+\frac{n(n-1)}{1.2}\Delta+\ldots+\Delta^{n-1}\right\}\Sigma_x(v)\ (\Delta x=1),$$

we obtain $\Sigma_{x+n}(v)-\Sigma_x(v)=\dfrac{1}{(x+1)^v}+\dfrac{1}{(x+2)^v}+\ldots+\dfrac{1}{(x+n)^v}$

$$=n\left\{\frac{1}{x+1}\,\frac{\phi_{v-1}(x,1)}{1}-\frac{n-1}{x+1\ x+2}\,\frac{\phi_{v-1}(x,2)}{2}\right.$$

$$+\frac{(n-1)(n-2)}{(x+1)(x+2)(x+3)}\,\frac{\phi_{v-1}(x,3)}{3}-\ldots$$

$$\left.+(-1)^{n-1}\frac{(n-1)(n-2)\ldots3.2.1}{(x+1)(x+2)\ldots(x+n)}\,\frac{\phi_{v-1}(x,n)}{n}\right\}\ldots\ldots\ldots\ldots (25),$$

$$\Sigma_x(v)+\Sigma_{x+1}(v)+\ldots+\Sigma_{x+n-1}(v)=n\,\Sigma_x(v)$$

$$+n(n-1)\left\{\frac{1}{x+1}\,\frac{\phi_{v-1}(x,1)}{1.2}-\frac{n-2}{(x+1)(x+2)}\,\frac{\phi_{v-1}(x,2)}{2.3}\right.$$

$$+\frac{(n-2)(n-3)}{(x+1)(x+2)(x+3)}\,\frac{\phi_{v-1}(x,3)}{3.4}-\ldots$$

$$\left.+\frac{(-1)^{n-2}(n-2)\ldots3.2.1}{(x+1)\ldots(x+n-1)}\,\frac{\phi_{v-1}(x,n-1)}{(n-1)n}\right\}\ldots\ldots\ldots\ldots\ldots\ldots\ldots (26),$$

and if $x=0$,

$$\frac{1}{1^v}+\frac{1}{2^v}+\ldots+\frac{1}{n^v}=\frac{n}{1}\,\frac{\phi_{v-1}(0,1)}{1}-\frac{n\,n-1}{1.2}\,\frac{\phi_{v-1}(0,2)}{2}$$

$$+\frac{n(n-1)(n-2)}{1.2.3}\,\frac{\phi_{v-1}(0,3)}{3}-\ldots+(-1)^{n-1}\frac{\phi_{v-1}(0,n)}{n}\ldots (27),$$

$$\Sigma_1(v)+\Sigma_2(v)+\ldots+\Sigma_n(v)=\frac{(n+1)n}{1.2}\,\frac{\phi_{v-1}(0,1)}{1}$$

$$-\frac{(n+1)n(n-1)}{1.2.3}\,\frac{\phi_{v-1}(0,2)}{2}+\ldots+(-1)^{n-1}\frac{\phi_{v-1}(0,n)}{n}\ldots (28).$$

Ex. Let $v=2$, $\dfrac{1}{(x+1)^2}+\dfrac{1}{(x+2)^2}+\ldots+\dfrac{1}{(x+n)^2}$

$$=n\left\{\frac{1}{x+1}\,\frac{\Sigma_{x+1}-\Sigma_x}{1}-\frac{(n-1)}{(x+1)(x+2)}\cdot\frac{\Sigma_{x+2}-\Sigma_x}{2}+\ldots\right.$$

$$\left.+(-1)^{n-1}\frac{(n-1)(n-2)\ldots3.2.1}{(x+1)(x+2)\ldots(x+n)}\,\frac{\Sigma_{x+n}-\Sigma_x}{n}\right\},$$

$$\text{and } \frac{1}{1^2}+\frac{1}{2^2}+\ldots+\frac{1}{n^2}=n\frac{\Sigma_1}{1}-\frac{n(n-1)}{1.2}\frac{\Sigma_2}{2}$$
$$+\frac{n(n-1)(n-2)}{1.2.3}\frac{\Sigma_3}{3}-\ldots+(-1)^{n-1}\frac{\Sigma_n}{n}.\text{*}$$

From (28), we have the formula (17), Art. 7.

Let $v=3$ and $x=0$, then

$$\frac{1}{1^3}+\frac{1}{2^3}+\frac{1}{3^3}+\ldots+\frac{1}{n^3}=n\Sigma_1-\frac{n(n-1)}{1.2}\frac{1}{2}\left(\frac{\Sigma_1}{1}+\frac{\Sigma_2}{2}\right)$$
$$+\frac{n(n-1)(n-2)}{1.2.3}\frac{1}{3}\left(\frac{\Sigma_1}{1}+\frac{\Sigma_2}{2}+\frac{\Sigma_3}{3}\right)+\ldots$$
$$+(-1)^{n-1}\frac{1}{n}\left(\frac{\Sigma_1}{1}+\frac{\Sigma_2}{2}+\frac{\Sigma_3}{3}+\ldots+\frac{\Sigma_n}{n}\right).$$

11. Also $\Sigma_x^{(m)}+\Sigma_{x+1}^{(m)}+\ldots+\Sigma_{x+n}^{(m)}$

$$=(n+1)\Sigma_x^m+\frac{\Gamma x+1}{\Gamma(x+m+1)}\left\{\frac{(n+1)n}{1.2}-\frac{(n+1)n(n-1)}{1.2.3}\frac{m}{x+m+1}\right.$$
$$+\frac{(n+1)n(n-1)(n-2)}{1.2.3.4}\frac{m(m+1)}{(x+m+1)(x+m+2)}-\ldots$$
$$\left.+(-1)^{n-1}\frac{m(m+1)\ldots(m+n-2)}{(x+m+1)\ldots(x+m+n-1)}\right\}\ldots\ldots\ (29),$$

and $\Sigma_1^{(m)}+\Sigma_2^{(m)}+\ldots+\Sigma_n^{(m)}$

$$=\frac{1}{\Gamma m}\left\{\frac{(n+1)n}{1.2}\frac{1}{m}-\frac{(n+1)n(n-1)}{1.2.3}\frac{1}{m+1}\right.$$
$$\left.+\frac{(n+1)n(n-1)(n-2)}{1.2.3.4}\frac{1}{m+2}-\ldots+(-1)^{n-1}\frac{1}{m+n-1}\right\}\ldots(30).$$

12. Let

$$\theta_1(x,n)=\Sigma_{x+n}-\Sigma_{x-1},$$
$$\theta_2(x,n)=\frac{1}{x+1}\theta_1(x,1)+\frac{1}{x+2}\theta_1(x,2)+\ldots+\frac{1}{x+n}\theta_1(x,n),$$
$$\theta_3(x,n)=\frac{1}{x+1}\theta_2(x,1)+\frac{1}{x+2}\theta_2(x,2)+\ldots+\frac{1}{x+n}\theta_2(x,n),$$

....

$$\theta_v(x,n)=\frac{1}{x+1}\theta_{v-1}(x,1)+\frac{1}{x+2}\theta_{v-1}(x,2)+\ldots+\frac{1}{x+n}\theta_{v-1}(x,n),$$

* Blissard, "On the Sums of Reciprocals," Art. 13, p. 257.

then from the equivalence

$$\Delta^{n-1}\frac{\Sigma_{x+1}}{(x+1)^v}=(-1)^{n-1}\left\{1-(n-1)(1+\Delta)+\frac{(n-1)(n-2)}{1.2}(1+\Delta)^2-\dots\right.$$

$$\left.+(-1)^{n-1}(1+\Delta)^{n-1}\right\}\frac{\Sigma_{x+1}}{(x+1)^v}\,(\Delta x=1),$$

we obtain $\theta_v(x,n)=\dfrac{\Gamma(x+n+1)}{\Gamma n\,\Gamma(x+1)}\left\{\dfrac{\Sigma_{x+1}}{(x+1)^v}-(n-1)\dfrac{\Sigma_{x+2}}{(x+2)^v}\right.$

$$\left.+\frac{(n-1)(n-2)}{1.2}\frac{\Sigma_{x+3}}{(x+3)^v}-\dots+(-1)^{n-1}\frac{\Sigma_{x+n}}{(x+n)^v}\right\}\dots\dots(31),$$

and $\theta_v(0,n)=\dfrac{n\Sigma_1}{1^{v-1}}-\dfrac{n(n-1)}{1.2}\dfrac{\Sigma_2}{2^{v-1}}$

$$+\frac{n(n-1)(n-2)}{1.2.3}\frac{\Sigma_3}{3^{v-1}}-\dots+(-1)^{n-1}\frac{\Sigma_n}{n^{v-1}}\dots\dots(32).$$

Ex. If $v=2$,

$$\frac{\Sigma_{x+1}}{x+1}+\frac{\Sigma_{x+2}-\Sigma_1}{x+2}+\frac{\Sigma_{x+3}-\Sigma_2}{x+3}+\dots+\frac{\Sigma_{x+n}-\Sigma_{n-1}}{x+n}$$

$$=\frac{\Gamma(x+n+1)}{\Gamma n\,\Gamma(x+1)}\left\{\frac{\Sigma_{x+1}}{(x+1)^2}-(n-1)\frac{\Sigma_{x+2}}{(x+2)^2}+\frac{n(n-1)}{1.2}\frac{\Sigma_{x+3}}{(x+3)^2}-\dots\right.$$

$$\left.+(-1)^{n-1}\frac{\Sigma_{x+n}}{(x+n)^2}\right\},$$

and $\qquad \Sigma_n(2)=n\dfrac{\Sigma_1}{1}-\dfrac{n(n-1)}{1.2}\dfrac{\Sigma_2}{2}+\dots+(-1)^{n-1}\dfrac{\Sigma_n}{n},$

(see Art. 10), if $v=3$, from (32),

$$\frac{\Sigma_1(2)}{1}+\frac{\Sigma_2(2)}{2}+\frac{\Sigma_3(2)}{3}+\dots+\frac{\Sigma_n(2)}{n}=\frac{n\Sigma_1}{1^2}-\frac{n(n-1)}{1.2}\frac{\Sigma_2}{2^2}$$

$$+\frac{n(n-1)(n-2)}{1.2.3}\frac{\Sigma_3}{3^2}-\dots+(-1)^{n-1}\frac{\Sigma_n}{n^2}.$$

13. The equivalence

$$(1+\Delta)^{n-1}\frac{\Sigma_{x+1}}{(x+1)^v}=\left\{1+(n-1)\Delta+\frac{(n-1)(n-2)}{1.2}\Delta^2+\dots\right.$$

$$\left.+\Delta^{n-1}\right\}\frac{\Sigma_{x+1}}{(x+1)^v}\,(\Delta x=1),$$

gives $\dfrac{\Sigma_{x+n}}{(x+n)^v} = \dfrac{1}{x+1}\theta_v(x, 1) - \dfrac{(n-1)}{(x+1)(x+2)}\theta_v(x, 2)$

$$+ \frac{(n-1)(n-2)}{(x+1)(x+2)(x+3)}\theta_v(x, 3) - \dots$$

$$+ (-1)^{n-1}\frac{(n-1)(n-2)\dots 3.2.1}{(x+1)(x+2)\dots(x+n)}\theta_v(x, n) \dots\dots\dots\dots\dots (33),$$

and $\dfrac{\Sigma_n}{n^v} = \dfrac{\theta_v(0, 1)}{1} - \dfrac{(n-1)}{1}\dfrac{\theta_v(0, 2)}{2}$

$$+ \frac{(n-1)(n-2)}{1.2}\frac{\theta_v(0, 3)}{3} - \dots + (-1)^{n-1}\frac{\theta_v(0, n)}{n} \dots (34).$$

Ex. $v = 3\ \dfrac{\Sigma_n}{n^3} = \dfrac{\Sigma_1(2)}{1} - \dfrac{(n-1)}{1}\dfrac{1}{2}\left(\dfrac{\Sigma_1(2)}{1} + \dfrac{\Sigma_2(2)}{2}\right)$

$$+ \frac{(n-1)(n-2)}{1.2}\frac{1}{3}\left\{\frac{\Sigma_1(2)}{1} + \frac{\Sigma_2(2)}{2} + \frac{\Sigma_3(2)}{3}\right\} - \dots,$$

$$+ (-1)^{n-1}\frac{1}{n}\left\{\frac{\Sigma_1(2)}{1} + \frac{\Sigma_2(2)}{2} + \frac{\Sigma_3(2)}{3} + \dots + \frac{\Sigma_n(2)}{n}\right\}.$$

14. By virtue of the equivalence

$$\{1 + (1+\Delta) + (1+\Delta)^2 + \dots + (1+\Delta)^{n-1}\}\frac{\Sigma_{x+1}}{(x+1)^v}$$

$$= \left\{n + \frac{n(n-1)}{1.2}\Delta + \frac{n(n-1)(n-2)}{1.2.3}\Delta^2 + \dots + \Delta^{n-1}\right\}\frac{\Sigma_{x+1}}{(x+1)^v},\ (\Delta x = 1),$$

we have $\dfrac{\Sigma_{x+1}}{(x+1)^v} + \dfrac{\Sigma_{x+2}}{(x+2)^v} + \dots + \dfrac{\Sigma_{x+n}}{(x+n)^v}$

$$= \frac{n}{x+1}\frac{\theta_v(x, 1)}{1} - \frac{n(n-1)}{(x+1)(x+2)}\frac{\theta_v(x, 2)}{2}$$

$$+ \frac{n(n-1)(n-2)}{(x+1)(x+2)(x+3)}\frac{\theta_v(x, 3)}{3} - \dots$$

$$+ (-1)^{n-1}\frac{n(n-1)\dots 3.2.1}{(x+1)(x+2)\dots(x+n)}\frac{\theta_v(x, n)}{n} \dots\dots\dots\dots\dots (35),$$

and $\dfrac{\Sigma_1}{1^v} + \dfrac{\Sigma_2}{2^v} + \dots + \dfrac{\Sigma_n}{n^v} = \dfrac{n}{1}\dfrac{\theta_v(0, 1)}{1} - \dfrac{n(n-1)}{1.2}\dfrac{\theta_v(0, 2)}{2}$

$$+ \frac{n(n-1)(n-2)}{1.2.3}\frac{\theta_v(0, 3)}{3} - \dots + (-1)^{n-1}\frac{\theta_v(0, n)}{n} \dots (36).$$

Ex. $v=2$, $\frac{\Sigma_1}{1^2}+\frac{\Sigma_2}{2^2}+\ldots+\frac{\Sigma_n}{n^2}$

$$=\frac{n}{1}\frac{\Sigma_1(2)}{1}-\frac{n(n-1)}{1.2}\frac{\Sigma_2(2)}{2}+\ldots+(-1)^{n-1}\frac{\Sigma_n(2)}{n},$$

$v=3$, $\frac{\Sigma_1}{1^3}+\frac{\Sigma_2}{2^3}+\ldots+\frac{\Sigma_n}{n^3}$

$$=\frac{n}{1}\frac{\Sigma_1(2)}{1}-\frac{n(n-1)}{1.2}\frac{1}{2}\left\{\frac{\Sigma_1(2)}{1}+\frac{\Sigma_2(2)}{2}\right\}+\ldots$$

$$+(-1)^{n-1}\frac{1}{n}\left\{\frac{\Sigma_1(2)}{1}+\frac{\Sigma_2(2)}{2}+\ldots+\frac{\Sigma_n(2)}{n}\right\}.$$

15. Let

$$\theta_1'(x,n)=\Sigma_x-\Sigma_{n-1},$$

$$\theta_2'(x,n)=\frac{1}{x+1}\theta_1'(x,1)+\frac{1}{x+2}\theta_1'(x,2)+\ldots+\frac{1}{x+n}\theta_1'(x,n),$$

.... &c.,

then since

$$(\alpha)\ \Delta^{n-1}\frac{\Sigma_x}{(x+1)^v}=(-1)^{n-1}\left\{1-(n-1)(1+\Delta)+\frac{(n-1)(n-2)}{1.2}(1+\Delta)^2-\ldots\right.$$

$$\left.+(-1)^{n-1}(1+\Delta)^{n-v}\right\}\frac{\Sigma_x}{(x+1)^v}\ (\Delta x=1).$$

$$(\beta)\ (1+\Delta)^n\frac{\Sigma_x}{(x+1)^v}=\left\{1+n\Delta+\frac{n(n-1)}{1.2}\Delta^2+\ldots+\Delta^n\right\}\frac{\Sigma_x}{(x+1)^v},$$

and (γ) $\{1+(1+\Delta)+(1+\Delta)^2+\ldots+(1+\Delta)^{n-2}\}\frac{\Sigma_x}{(x+1)^v}$

$$=\left\{(n-1)+\frac{(n-1)(n-2)}{1.2}\Delta+\ldots+\Delta^{n-2}\right\}\frac{\Sigma_x}{(x+1)^v},$$

we find from (α)

$$\theta_v'(x,n)=\frac{\Gamma(x+n+1)}{\Gamma n\,\Gamma(x+1)}\left\{\frac{\Sigma_x}{(x+1)^v}-(n-1)\frac{\Sigma_{x+1}}{(x+2)^v}\right.$$

$$\left.+\frac{(n-1)(n-2)}{1.2}\frac{\Sigma_{x+2}}{(x+3)^v}-\ldots+(-1)^{n-1}\frac{\Sigma_{x+n-1}}{(x+n)^v}\right\}\ldots(37),$$

$$\text{and } \theta_v'(0, n) = -\left\{\frac{n(n-1)}{1.2}\frac{\Sigma_1}{2^{v-1}} - \frac{n(n-1)(n-2)}{1.2.3}\frac{\Sigma_2}{3^{v-1}} + \dots \right.$$

$$\left. + (-1)^{n-2}\frac{\Sigma_{n-1}}{n^{v-1}}\right\} \dots\dots\dots\dots\dots (38).$$

Ex. If $v = 2$,

$$\frac{\Sigma_1}{2} + \frac{\Sigma_2}{3} + \frac{\Sigma_3}{4} + \dots + \frac{\Sigma_{n-1}}{n}$$

$$= \frac{n(n-1)}{1.2}\frac{\Sigma_1}{2} - \frac{n(n-1)(n-2)}{1.2.3}\frac{\Sigma_2}{3} + \dots + (-1)^{n-2}\frac{\Sigma_{n-1}}{n};$$

from (β)

$$\frac{\Sigma_{x+n}}{(x+n+1)^v} = \frac{1}{x+1}\theta_v'(x, 1) - \frac{n}{(x+1)(x+2)}\theta_v'(x, 2)$$

$$+ \frac{n(n-1)}{(x+1)(x+2)(x+3)}\theta_v'(x, 3) - \dots$$

$$+ (-1)^n \frac{n(n-1)\dots 3.2.1}{(x+1)(x+2)\dots(x+n+1)}\theta_v'(x, n+1) \dots\dots (39),$$

$$\text{and } \frac{\Sigma_n}{(n+1)^v} = -\left\{\frac{n}{1}\frac{\theta_v'(0, 2)}{2} - \frac{n(n-1)}{1.2}\frac{\theta_v'(0, 3)}{3} + \dots \right.$$

$$\left. + (-1)^{n-1}\frac{\theta_v'(0, n+1)}{n+1}\right\} \dots\dots\dots\dots (40),$$

and from (γ)

$$\frac{\Sigma_x}{(x+1)^v} + \frac{\Sigma_{x+1}}{(x+2)^v} + \frac{\Sigma_{x+2}}{(x+3)^v} + \dots + \frac{\Sigma_{x+n-2}}{(x+n-1)^v}$$

$$= \frac{n-1}{x+1}\frac{\theta_v'(x, 1)}{1} - \frac{(n-1)(n-2)}{(x+1)(x+2)}\frac{\theta_v'(x, 2)}{2} + \dots$$

$$+ (-1)^{n-2}\frac{(n-1)(n-2)\dots 2.1}{(x+1)\dots(x+n-1)}\frac{\theta_v'(x, n-1)}{n-1} \dots\dots (41),$$

and if $x = 1$,

$$\frac{\Sigma_1}{2^v} + \frac{\Sigma_2}{3^v} + \dots + \frac{\Sigma_{n-1}}{n^v} = \frac{n-1}{1}\frac{\theta_v'(1, 1)}{1.2} - \frac{(n-1)(n-2)}{1.2}\frac{\theta_v'(1, 2)}{2.3}$$

$$+ \frac{(n-1)(n-2)(n-3)}{1.2.3}\frac{\theta_v'(1, 3)}{3.4} - \dots + (-1)^{n-2}\frac{\theta_v'(1, n-1)}{(n-1)n} \dots (42).$$

Ex. Let $v=1$, then

$$\frac{\Sigma_1}{2}+\frac{\Sigma_2}{3}+\frac{\Sigma_3}{4}+\ldots+\frac{\Sigma_{n-1}}{n}$$

$$=(n-1)\frac{1}{1.2}-\frac{(n-1)(n-2)}{1.2}\frac{1-\Sigma_1}{2.3}$$

$$+\frac{(n-1)(n-2)(n-3)}{1.2.3}\frac{1-\Sigma_2}{3.4}-\ldots+(-1)^{n-2}\frac{1-\Sigma_{n-2}}{(n-1)n}$$

$$=(n-1)\frac{1}{1.2}-\frac{(n-1)(n-2)}{1.2}\frac{1}{2.3}+\frac{(n-1)(n-2)(n-3)}{1.2.3}\frac{1}{3.4}-\ldots$$

$$+(-1)^{n-2}\frac{1}{(n-1)n}+\frac{(n-1)(n-2)}{1.2}\frac{\Sigma_1}{2.3}$$

$$-\frac{(n-1)(n-2)(n-3)}{1.2.3}\frac{\Sigma_2}{3.4}+\ldots+(-1)^{n-3}\frac{\Sigma_{n-2}}{(n-1)n}.$$

But $(n-1)\frac{1}{1.2}-\frac{(n-1)(n-2)}{1.2}\frac{1}{2.3}+\ldots+(-1)^{n-2}\frac{1}{(n-1)n}=\Sigma_n-1$,

(see formula 10, Art. 4); therefore

$$\frac{\Sigma_1}{2}+\frac{\Sigma_2}{3}+\ldots+\frac{\Sigma_{n-1}}{n}=\Sigma_n-1+\frac{(n-1)(n-2)}{1.2}\frac{\Sigma_1}{2.3}$$

$$-\frac{(n-1)(n-2)(n-3)}{1.2.3}\frac{\Sigma_2}{3.4}+\ldots+(-1)^{n-3}\frac{\Sigma_{n-2}}{(n-1)n}.^*$$

16. From the formulæ

$$\left\{1-n(1+\Delta)^2+\frac{n(n-1)}{1.2}(1+\Delta)^4\right.$$

$$\left.-\frac{n(n-1)(n-2)}{1.2.3}(1+\Delta)^6+\ldots+(-1)^n(1+\Delta)^{2n}\right\}\Sigma_x(v)$$

$$=(-1)^n 2^n\left\{1+n\frac{\Delta}{2}+\frac{n(n-1)}{1.2}\frac{\Delta^2}{2^2}+\ldots+\frac{\Delta^n}{2^n}\right\}\Delta^n\Sigma_x(v)\ (\Delta x=1),$$

and $\left\{1-n(1+\Delta)^2+\frac{n(n-1)}{1.2}(1+\Delta)^4-\ldots+(-1)^n(1+\Delta)^{2n}\right\}\Sigma_x^{(m)}$

$$=(-1)^n 2^n\left\{1+n\frac{\Delta}{2}+\frac{n(n-1)}{1.2}\frac{\Delta^2}{2^2}+\ldots+\frac{\Delta^n}{2^n}\right\}\Delta^n\Sigma_x^{(m)},$$

* Blissard, "On the Sums of Reciprocals," Art. 12, p. 256, (3).

we have $\Sigma_x(v) - n\,\Sigma_{x+2}(v) + \frac{n(n-1)}{1.2}\,\Sigma_{x+4}(v)$

$$- \frac{n(n-1)(n-2)}{1.2.3}\,\Sigma_{x+6}(v) + \ldots + (-1)^n\,\Sigma_{x+2n}(v)$$

$$= -2^n\,\frac{\Gamma n\,\Gamma(x+1)}{\Gamma(x+n+1)}\left\{\phi_{v-1}(x, n) - \frac{n}{1}\,\frac{n}{x+n+1}\,\frac{\phi_{v-1}(x, n+1)}{2}\right.$$

$$+ \frac{n(n-1)}{1.2}\,\frac{n(n+1)}{(x+n+1)(x+n+2)}\,\frac{\phi_{v-1}(x, n+2)}{2^2} - \ldots$$

$$\left. + (-1)^n\,\frac{n(n+1)(n+2)\ldots(2n-1)}{(x+n+1)(x+n+2)\ldots(x+2n)}\,\frac{\phi_{v-1}(x, 2n)}{2^n}\right\}\ldots(43),$$

and $\Sigma_x^{(m)} - n\,\Sigma_{x+2}^{(m)} + \frac{n(n-1)}{1.2}\,\Sigma_{x+4}^{(m)} - \ldots + (-1)^n\,\Sigma_{x+2n}^{(m)}$

$$= -2^n\,\frac{\Gamma(m+n-1)\,\Gamma(x+1)}{\Gamma(x+m+n)}\left\{1 - \frac{n(m+n-1)}{x+m+n}\,\frac{1}{2}\right.$$

$$+ \frac{n(n-1)}{1.2}\,\frac{(m+n-1)(m+n)}{(x+m+n)(x+m+n+1)}\,\frac{1}{2^2} - \ldots$$

$$\left. + (-1)^n\,\frac{(m+n-1)(m+n)\ldots(m+2n-2)}{(x+m+n)(x+m+n+1)\ldots(x+m+2n-1)}\,\frac{1}{2^n}\right\}\ldots(44).$$

In the same way we can obtain the following results:

$$\frac{\Sigma_{x+1}}{(x+1)^v} - n\,\frac{\Sigma_{x+3}}{(x+3)^v} + \frac{n(n-1)}{1.2}\,\frac{\Sigma_{x+5}}{(x+5)^v} - \ldots + (-1)^n\,\frac{\Sigma_{x+2n+1}}{(x+2n+1)^v}$$

$$= 2^n\,\frac{\Gamma(n+1)\,\Gamma(x+1)}{\Gamma(x+n+2)}\left\{\theta_v(x, n+1) - \frac{n}{1}\,\frac{n+1}{(x+n+2)}\,\frac{\theta_v(x, n+2)}{2} + \ldots\right.$$

$$\left. + (-1)^n\,\frac{(n+1)(n+2)\ldots 2n}{(x+n+2)\ldots(x+2n+1)}\,\frac{\theta_v(x, 2n+1)}{2^n}\right\}\ldots\ldots\ldots(45),$$

and $\frac{\Sigma_x}{(x+1)^v} - n\,\frac{\Sigma_{x+2}}{(x+3)^v} + \frac{n(n-1)}{1.2}\,\frac{\Sigma_{x+4}}{(x+5)^v} - \ldots + (-1)^n\,\frac{\Sigma_{x+2n}}{(x+2n+1)^v}$

$$= 2^n\,\frac{\Gamma(n+1)\,\Gamma(x+1)}{\Gamma(x+n+2)}\left\{\theta_v'(x, n+1) - \frac{n}{1}\,\frac{n+1}{x+n+2}\,\frac{\theta_v'(x, n+2)}{2} + \ldots\right.$$

$$\left. + (-1)^n\,\frac{(n+1)(n+2)\ldots 2n}{(x+n+2)\ldots(x+2n+1)}\,\frac{\theta_v'(x, 2n+1)}{2^n}\right\}\ldots\ldots\ldots(46).$$

Ex. $v=1$, $\Sigma_x - n\,\Sigma_{x+2} + \frac{n(n-1)}{1.2}\,\Sigma_{x+4} - \ldots + (-1)^n\,\Sigma_{x+2n}$

$$= -2^n \frac{\Gamma(x+1)\,\Gamma n}{\Gamma(x+n+1)} \left\{1 - n\frac{n}{x+n+1}\,\frac{1}{2} + \frac{n(n-1)}{1.2}\,\frac{n(n+1)}{(x+n+1)(x+n+2)}\,\frac{1}{2^2} - \ldots\right\}.^*$$

17. From the equivalence

$$(-1)^{s+n}\left\{1-(n+s)(1+\Delta)+\frac{(n+s)(n+s-1)}{1.2}(1+\Delta)^2-\ldots + (-1)^{n+s}(1+\Delta)^{n+s}\right\}\Sigma_x(v)\;(\Delta x=1)$$

$$=(-1)^s\left\{1-s(1+\Delta)+\frac{s(s-1)}{1.2}(1+\Delta)^2-\ldots + (-1)^s(1+\Delta)^s\right\}\Delta^n\,\Sigma_x(v)=\Delta^{n+s}\,\Sigma_x(v),$$

we have

$$\Sigma_x(v)-(n+s)\,\Sigma_{x+1}(v)+\frac{(n+s)(n+s-1)}{1.2}\,\Sigma_{x+2}(v)-\ldots+(-1)^{n+s}\,\Sigma_{x+n+s}(v)$$

$$= -\frac{\Gamma n\,\Gamma(x+1)}{\Gamma(x+n+1)}\left\{\phi_{v-1}(x, n) - s\phi_{v-1}(x+1, n)\,\frac{x+1}{(x+n+1)} + \frac{s(s-1)}{1.2}\,\frac{(x+1)(x+2)}{(x+n+1)(x+n+2)}\,\phi_{v-1}(x+2, n) - \ldots + (-1)^s\,\frac{(x+1)(x+2)\ldots(x+s)}{(x+n+1)\ldots(x+n+s)}\,\phi_{v-1}(x+s, n)\right\}$$

$$= -\frac{\Gamma(n+s)\,\Gamma(x+1)}{\Gamma(x+n+s+1)}\,\phi_{v-1}(x, n+s) \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\; (47).$$

Hence, supposing $v=1$, we obtain Mr. Blissard's formula†

$$1 - s\frac{x+1}{x+n+1} + \frac{s(s-1)}{1.2}\,\frac{(x+1)(x+2)}{(x+n+1)(x+n+2)} - \ldots = \frac{\Gamma(n+s)\,\Gamma(x+n+1)}{\Gamma n\,\Gamma(x+n+s+1)} \ldots\ldots\ldots\ldots (48).$$

For the functions $\frac{\Sigma_{x+1}}{(x+1)^v}$, $\theta(x, n)$ and $\frac{\Sigma_x}{(x+1)^v}$, $\theta'(x, n)$ we may obtain the similar relations.

* Blissard, "On the Sums of Reciprocals," Art. 12, p. 256, (2).

† "On the Generalization of Certain Formulæ investigated by Mr. Walton," *Quarterly Journal*, 1863, No. 22, Art. 3, p. 168.

18. Supposing $\Delta x = h$ in the equivalence

$$\Delta^n \Sigma_x = (-1)^n \left\{1 - n(1+\Delta) + \frac{n(n-1)}{1.2}(1+\Delta)^2 - \ldots + (-1)^n (1+\Delta)^n\right\} \Sigma_x,$$

we find

$$h^{n-1}\, 1.2\ldots(n-1) \left\{\frac{1}{(x+1)(x+h+1)\ldots[x+(n-1)h+1]}\right.$$

$$+ \frac{1}{(x+2)(x+h+2)\ldots[x+(n-1)h+2]}$$

$$\left.+ \frac{1}{(x+3)(x+h+3)\ldots[x+(n-1)h+3]} + \ldots + \frac{1}{(x+h)(x+2h)\ldots(x+nh)}\right\}$$

$$= -\left\{\Sigma_x - n\,\Sigma_{x+h} + \frac{n(n-1)}{1.2}\Sigma_{x+2h} - \ldots + (-1)^n \Sigma_{x+nh}\right\} \ldots\ldots (49).$$

Hence, if $x = 0$,

$$h^{n-1} 1.2.3\ldots(n-1) \left\{\frac{1}{1(h+1)\ldots[(n-1)h+1]} + \frac{1}{2(h+2)\ldots[(n-1)h+2]}\right.$$

$$+ \frac{1}{3(h+3)\ldots[(n-1)h+3]} + \ldots + \frac{1}{(h-1)(2h-1)\ldots(nh-1)}$$

$$\left.+ \frac{1}{nh} = n\Sigma_h - \frac{n(n-1)}{1.2}\Sigma_{2h} + \ldots + (-1)^{n-1}\Sigma_{nh}\right\} \ldots\ldots. (50).$$

Ex. Let $h = 2$,

$$\frac{2.4\ldots(2n-2)}{1.3\ldots(2n-1)} + \frac{1}{2n} = n\,\Sigma_2 - \frac{n(n-1)}{1.2}\Sigma_4 + \ldots + (-1)^{n-1}\Sigma_{2n}.^*$$

19. The equivalence

$$(-1)^n \left\{1 - n(1+\Delta) + \frac{n(n-1)}{1.2}(1+\Delta)^2 - \ldots\right.$$

$$\left. + (-1)^n (1+\Delta)^n\right\} \frac{\Sigma_{x+1}}{x} = \Delta^n \frac{\Sigma_{n+1}}{x} \; (\Delta x = 1),$$

gives $$\frac{\Sigma_{x+1}}{x} - n\frac{\Sigma_{x+2}}{x+1} + \frac{n(n-1)}{1.2}\frac{\Sigma_{x+3}}{x+2} - \ldots + (-1)^n \frac{\Sigma_{x+n+1}}{x+n}$$

$$= \frac{\Gamma(n+1)\Gamma(x+2)}{\Gamma(x+n+1)} \left\{\frac{\Sigma_{x+1}}{x(x+1)} - \left[\frac{1}{1(x+1)(x+2)} + \frac{1}{2(x+2)(x+3)} + \ldots\right.\right.$$

$$\left.\left. + \frac{1}{n(x+n)(x+n+1)}\right]\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (51).$$

* Blissard, "On the Sums of Reciprocals," Art. 12, p. 257, (4).

Hence, if $x=0$,

$$n\frac{\Sigma_2}{1}-\frac{n(n-1)}{1.2}\frac{\Sigma_3}{2}+\ldots+(-1)^{n-1}\frac{\Sigma_{n+1}}{n}$$

$$=\Sigma_n+\frac{1}{1^2 2}+\frac{1}{2^2 3}+\frac{1}{3^2 4}+\ldots+\frac{1}{n^2\,\overline{n+1}}\ \ldots\ldots\ (52).$$

20. Making $x=0$ in the formula

$$(-1)^n\left\{1-n(1+\Delta)+\frac{n(n-1)}{1.2}(1+\Delta)^2-\ldots\right.$$

$$\left.+(-1)^n(1+\Delta)^n\right\}\frac{1}{x2^x}=\Delta^n\frac{1}{x2^x}\ (\Delta x=1),$$

we have $\Sigma_n-\Sigma'_n=n\dfrac{1}{1.2}-\dfrac{n(n-1)}{1.2}\dfrac{1}{22^2}$

$$+\frac{n(n-1)(n-2)}{1.2.3}\frac{1}{32^3}-\ldots+(-1)^{n-1}\frac{1}{n2^n}\ \ldots\ldots\ (53),$$

where $\Sigma_n'=\dfrac{1}{1.2}+\dfrac{1}{22^2}+\dfrac{1}{32^3}+\ldots+\dfrac{1}{n2^n}$,

(Mr. Blissard's notation).

21. From the equivalence

$$\left(\frac{s}{1+s}\right)^n\left\{1+n\frac{s}{1+s}(1+\Delta)+\frac{n(n+1)}{1.2}\left(\frac{s}{1+s}\right)^2(1+\Delta)^2+\ldots\right\}\Delta^m\Sigma_x(v)$$

$$=(-1)^n\left(\frac{s}{1+s}\right)^{n-x}\Delta^{-n}\left\{\left(\frac{s}{1+s}\right)^x\Delta^m\,\Sigma_x(v)\right\}(\Delta x=1),$$

we obtain

$$\left(\frac{s}{1+s}\right)^n\phi_{v-1}(x,m)+\frac{n}{1}\frac{x+1}{x+m+1}\left(\frac{s}{1+s}\right)^{n+1}\phi_{v-1}(x+1,m)$$

$$+\frac{n(n+1)}{1.2}\frac{(x+1)(x+2)}{(x+m+1)(x+m+2)}\left(\frac{s}{1+s}\right)^{n+2}\phi_{v-1}(x+2,m)+\ldots$$

$$=s^n\phi_{v-1}(x,m)-\frac{n}{1}\frac{m}{x+m+1}s^{n+1}\phi_{v-1}(x,m+1)$$

$$+\frac{n(n+1)}{1.2}\frac{m(m+1)}{(x+m+1)(x+m+2)}s^{n+2}\phi_{v-1}(x,m+2)+\ldots\ \ldots\ (54).^*$$

Hence, if $v=1$, we have Mr. Blissard's remarkable formula (see his paper "On the Generalization of Certain

* It is easy to shew that $C_1=0$, $C_2=0$, ... $C_n=0$, C_1, C_2 ... C_n being the arbitrary constants.

Formulæ investigated by Mr. Walton," *Quarterly Journal*, 1863, No. 22, p. 173)

$$\left(\frac{s}{1+s}\right)^n+\frac{n}{1}\frac{x+1}{x+m+1}\left(\frac{s}{1+s}\right)^{n+1}$$
$$+\frac{n(n+1)}{1.2}\frac{(x+1)(x+2)}{(x+m+1)(x+m+2)}\left(\frac{s}{1+s}\right)^{n+2}+\ldots$$
$$=s^n-\frac{n}{1}\frac{m}{x+m+1}s^{n+1}$$
$$+\frac{n(n+1)}{1.2}\frac{m(m+1)}{(x+m+1)(x+m+2)}s^{n+2}-\ldots\ldots\ldots \quad (55).$$

Changing s into Δ and $x+1$ into s,

$$\left(\frac{\Delta}{1+\Delta}\right)^n+\frac{n}{1}\frac{s}{s+m}\left(\frac{\Delta}{1+\Delta}\right)^{n+1}$$
$$+\frac{n(n+1)}{1.2}\frac{s(s+1)}{(s+m)(s+m+1)}\left(\frac{\Delta}{1+\Delta}\right)^{n+2}+\ldots$$
$$=\Delta^n-\frac{n}{1}\frac{m}{s+m}\Delta^{n+1}$$
$$+\frac{n(n+1)}{1.2}\frac{m(m+1)}{(s+m)(s+m+1)}\Delta^{n+2}-\ldots\ldots\ldots\ldots\ldots \quad (56).$$

and if $m=0$,

$$\Delta^n=\left(\frac{\Delta}{1+\Delta}\right)^n+n\left(\frac{\Delta}{1+\Delta}\right)^{n+1}+\frac{n(n+1)}{1.2}\left(\frac{\Delta}{1+\Delta}\right)^{n+2}+\ldots\ldots(57),*$$

* Several interesting results may be obtained by the aid of the formula (57). For example, from the equivalence

$$\Delta^n x(x-1)(x-2)\ldots(x-2n+1)=\left\{\left(\frac{\Delta}{1+\Delta}\right)^n+\frac{n}{1}\left(\frac{\Delta}{1+\Delta}\right)^{n+1}\right.$$
$$\left.+\frac{n(n+1)}{1.2}\left(\frac{\Delta}{1+\Delta}\right)^{n+2}+\ldots\right\}x(x-1)(x-2)\ldots(x+2n+1)\;(\Delta x=1),$$

making $x=2n$, we have (n positive integer).

$$(a)\quad 1+n+\frac{n(n+1)}{1.2}+\frac{n(n+1)(n+2)}{1.2.3}+\ldots+\frac{n\ldots(2n-1)}{1.2\ldots n}$$
$$=\sum_{i=1}^{i=n}\frac{\Gamma(n+i)}{\Gamma n\,\Gamma(i+1)}=\frac{\Gamma(2n+1)}{\{\Gamma(n+1)\}^2}.$$

Also from the equivalence

$$\Delta^n x(x-1)(x-2)\ldots(x-q-n+p+1)=\left\{\left(\frac{\Delta}{1+\Delta}\right)^n+\frac{n}{1}\left(\frac{\Delta}{1+\Delta}\right)^{n+1}\right.$$
$$\left.+\frac{n(n+1)}{1.2}\left(\frac{\Delta}{1+\Delta}\right)^{n+2}+\ldots\right\}x(x-1)(x-2)\ldots(x-q-n+p+1),$$

the well known symbolic formula of the Calculus of Finite Differences.

Putting $-\theta(1+\Delta)$ for Δ in (56), we have

$$\left[\frac{(1+\Delta)\theta}{1-\theta(1+\Delta)}\right]^n - \frac{n}{1}\,\frac{s}{s+m}\left[\frac{(1+\Delta)\theta}{1-\theta(1+\Delta)}\right]^{n+1} + \frac{n(n+1)}{1.2}\,\frac{s(s+1)}{(s+m)(s+m+1)}\left[\frac{(1+\Delta)\theta}{1-\theta(1+\Delta)}\right]^{n+2} - \dots$$

$$= [\theta(1+\Delta)]^n + \frac{n}{1}\,\frac{m}{s+m}[\theta(1+\Delta)]^{n+1} + \frac{n(n+1)}{1.2}\,\frac{m(m+1)}{(s+m)(s+m+1)}[\theta(1+\Delta)]^{n+2} + \dots .$$

Let each member of this formula operate on $\frac{\Gamma(n)}{\Gamma(x)}$, and we obtain, supposing $x=0$, the expansion of ε^θ

$$\varepsilon^\theta = \frac{1+\frac{m}{s+m}\frac{\theta}{1}+\frac{m(m+1)}{(s+m)(s+m+1)}\frac{\theta^2}{1.2}+\dots}{1-\frac{s}{s+m}\frac{\theta}{1}+\frac{s(s+1)}{(s+m)(s+m+1)}\frac{\theta^2}{1.2}-\dots},$$

(Blissard, the same paper, p. 178), since

$$\left\{[1-\theta(1+\Delta)]^{-(n+t)}\frac{\Gamma n}{\Gamma(x+n+t)}\right\}_{x=0} = \frac{\varepsilon^\theta}{n(n+1)\dots(n+t-1)}.$$

supposing $x = q$, we find (n, p, q positive integers)

$$\frac{\Gamma(q-n+1)}{\Gamma(q-p+1)} + n\frac{\Gamma(q-n)}{\Gamma(q-p)} + \frac{n(n+1)}{1.2}\frac{\Gamma(q-n-1)}{\Gamma(q-p-1)} + \dots + \frac{n(n+1)(n+2)\dots(n+q-p-1)}{1.2.3\dots(q-p)}\frac{\Gamma(p-n+1)}{\Gamma 1}$$

$$= \frac{\Gamma(q+1)\,\Gamma(p-n+1)}{\Gamma(p+1)\,\Gamma(q-p+1)}\ (q \geqq p \geqq n).$$

Hence, if $n = 1$,

$$\frac{\Gamma q}{\Gamma(q-p+1)} + \frac{\Gamma(q-1)}{\Gamma(q-p)} + \frac{\Gamma(q-2)}{\Gamma(q-p-1)} + \dots + \frac{\Gamma p}{\Gamma 1} = \frac{\Gamma(q+1)}{p\,\Gamma(q-p+1)},$$

and if $p = n$, $q = 2n$, then we have the formula (α).

Minsk,
Dec. 26, 1865.

(To be continued.)

INVESTIGATION OF THE ENVELOPE OF THE STRAIGHT LINE JOINING THE FEET OF THE PERPENDICULARS LET FALL ON THE SIDES OF A TRIANGLE FROM ANY POSITION IN THE CIRCUMFERENCE OF THE CIRCUMSCRIBED CIRCLE.

By N. M. Ferrers.

The problem stated in the title of this paper is discussed by Mr. Greer, in a paper published in this *Journal*, t. VII., p. 70. It is there spoken of as a problem of some difficulty, and it certainly does not yield very readily to the application of trilinear coordinates. The following investigation, in which it is proved by the Cartesian method, that the required envelope is a well-known curve,—in fact, a three cusped hypocycloid, of which the centre of the nine-point circle of the triangle is the centre, may therefore not be without interest.

Let ABC be the triangle, O the centre of the circumscribing circle, S the point on its circumference, from which the perpendiculars SP, SQ, SR are let fall on the sides BC, CA, AB respectively. Let G be the centre of gravity of the triangle, O' the centre of the nine-point circle. Then we know that G lies on the straight line OO', and divides it so that OG is to GO' as two to one.

Let a be the radius of the circle, and first take O as origin. Let the radii through A, B, C, S be inclined to the axis of x at the angles α, β, γ, θ respectively, and choose the axis of x so that $\alpha+\beta+\gamma=0$.

The coordinates of G are

$$\frac{a}{3}(\cos\alpha+\cos\beta+\cos\gamma),\ \frac{a}{3}(\sin\alpha+\sin\beta+\sin\gamma),$$

hence those of O are

$$\frac{a}{2}(\cos\alpha+\cos\beta+\cos\gamma),\ \frac{a}{2}(\sin\alpha+\sin\beta+\sin\gamma).$$

And the equation of BC is

$$x\cos\frac{\beta+\gamma}{2}+y\sin\frac{\beta+\gamma}{2}=a\cos\frac{\beta-\gamma}{2},$$

i.e. $$x\cos\frac{\alpha}{2}\quad -y\sin\frac{\alpha}{2}\quad =a\cos\frac{\beta-\gamma}{2}.$$

Hence, that of SP is

$$x \sin\frac{\alpha}{2} + y \cos\frac{\alpha}{2} = a \sin\left(\frac{\alpha}{2} + \theta\right).$$

Therefore the abscissa of P is

$$a\left\{\cos\frac{\alpha}{2}\cos\frac{\beta-\gamma}{2} + \sin\frac{\alpha}{2}\sin\left(\frac{\alpha}{2}+\theta\right)\right\},$$

or $\quad \frac{1}{2}a\left\{\cos\frac{\alpha+\beta-\gamma}{2} + \cos\frac{\alpha-\beta+\gamma}{2} + \cos\theta - \cos(\alpha+\theta)\right\},$

or $\quad \frac{1}{2}a\{\cos\beta + \cos\gamma + \cos\theta - \cos(\alpha+\theta)\}.$

Similarly the ordinate is

$$\tfrac{1}{2}a\{\sin\beta + \sin\gamma + \sin\theta + \sin(\alpha+\theta)\}.$$

Now, transfer the origin to O', then the coordinates of P become

$$\tfrac{1}{2}a\{\cos\theta - \cos\alpha - \cos(\alpha+\theta)\},\ \ \tfrac{1}{2}a(\sin\theta - \sin\alpha + \sin(\alpha+\theta)\}.$$

Similarly, those of Q are

$$\tfrac{1}{2}a\{\cos\theta - \cos\beta - \cos(\beta+\theta)\},\ \ \tfrac{1}{2}a\{\sin\theta - \sin\beta + \sin(\beta+\theta)\},$$

and of R,

$$\tfrac{1}{2}a\{\cos\theta - \cos\gamma - \cos(\gamma+\theta)\},\ \ \tfrac{1}{2}a\{\sin\theta - \sin\gamma + \sin(\gamma+\theta)\}.$$

Therefore, the equation of PQR, is

$$\frac{x - \frac{1}{2}a\{\cos\theta - \cos\alpha - \cos(\alpha+\theta)\}}{\cos\gamma - \cos\beta + \cos(\gamma+\theta) - \cos(\beta+\theta)}$$

$$= \frac{y - \frac{1}{2}a\{\sin\theta - \sin\alpha + \sin(\alpha+\theta)\}}{\sin\gamma - \sin\beta - \sin(\gamma+\theta) + \sin(\beta+\theta)},$$

which may be reduced to

$$\frac{x - \frac{1}{2}a\{\cos\theta - \cos\alpha - \cos(\alpha+\theta)\}}{\cos\left(\gamma+\frac{\theta}{2}\right)\cos\frac{\theta}{2} - \cos\left(\beta+\frac{\theta}{2}\right)\cos\frac{\theta}{2}}$$

$$= \frac{y - \frac{1}{2}a\{\sin\theta - \sin\alpha + \sin(\alpha+\theta)\}}{\cos\left(\beta+\frac{\theta}{2}\right)\sin\frac{\theta}{2} - \cos\left(\gamma+\frac{\theta}{2}\right)\sin\frac{\theta}{2}},$$

or
$$\frac{x - \tfrac{1}{2}a\{\cos\theta - \cos\alpha - \cos(\alpha+0)\}}{\cos\frac{\theta}{2}}$$
$$= \frac{y - \tfrac{1}{2}a\{\sin\theta - \sin\alpha + \sin(\alpha+\theta)\}}{-\sin\frac{\theta}{2}},$$

or $x\sin\frac{\theta}{2} + y\cos\frac{\theta}{2}$

$$= \tfrac{1}{2}a\cos\frac{\theta}{2}\{\sin\theta - \sin\alpha + \sin(\alpha+\theta)\} + \tfrac{1}{2}a\sin\frac{\theta}{2}\{\cos\theta - \cos\alpha - \cos(\alpha+\theta)\}$$
$$= \tfrac{1}{2}a\sin\frac{3\theta}{2} - \tfrac{1}{2}a\sin\left(\frac{\theta}{2}+\alpha\right) + \tfrac{1}{2}a\sin\left(\frac{\theta}{2}+\alpha\right)$$
$$= \tfrac{1}{2}a\sin\frac{3\theta}{2}.$$

Hence, if p be the length of the perpendicular on PQR from O', ϕ its inclination to the axis of y, we see that

$$p = \tfrac{1}{2}a\sin 3\phi,$$

shewing that the envelope of the line is a three-cusped hypocycloid, in which the radius of the fixed circle is $\frac{3a}{2}$, and in which the axis of x is a tangent at a cusp.

Gonville and Caius College,
August 11, 1866.

ON THE CONICS WHICH PASS THROUGH TWO GIVEN POINTS AND TOUCH TWO GIVEN LINES.

By Professor Cayley.

Let $x=0$, $y=0$ be the equations of the given lines; $z=0$ the equation of the line joining the given points. We may, to fix the ideas, imagine the implicit constants so determined that $x+y+z=0$ shall be the equation of the line infinity.

Take $x - my = 0$, $x - ny = 0$ as the equations of the lines which by their intersection with $z = 0$ determine the given points. The equation of the conic is

$$\{\sqrt{(m)} + \sqrt{(n)}\} \sqrt{(xy)} = x + y\sqrt{(mn)} + \gamma z,$$

or, what is the same thing,

$$(x - my)(x - ny) + 2\{x + y\sqrt{(mn)}\}\gamma z + \gamma^2 z^2 = 0,$$

so that there are two distinct series of conics according as $\sqrt{(mn)}$ is taken with the positive or the negative sign.

The equation of the chord of contact is

$$x + y\sqrt{(mn)} + \gamma z = 0,$$

which meets $z = 0$ in the point $\{x + y\sqrt{(mn)} = 0,\ z = 0\}$ that is in one of the centres of the involution formed by the lines $(x = 0,\ y = 0)(x - my = 0,\ x - ny = 0)$. It is to be observed that the conic is only real when mn is positive, that is (the lines and points being each real) the two points must be situate in the same region or in opposite regions of the four regions formed by the two lines; there are however other real cases; *e.g.* if the lines $x = 0$, $y = 0$ are real, but the quantities m, n are conjugate imaginaries; included in this we have the circles which touch two real lines.

To fix the ideas I take m and n each positive and $mn > 1$; also I attend first to the series where $\sqrt{(mn)}$ is taken positively. At the points where the conic meets infinity, we have

$$\{\sqrt{(m)} + \sqrt{(n)}\} \sqrt{(xy)} = x + y\sqrt{(mn)} - \gamma(x + y),$$

which gives two coincident points, that is the conic is a parabola, if

$$(1 - \gamma)\{\sqrt{(mn)} - \gamma\} = \tfrac{1}{4}\{\sqrt{(m)} + \sqrt{(n)}\}^2,$$

that is $\quad \gamma^2 - \gamma\{1 + \sqrt{(mn)}\} = \tfrac{1}{4}\{\sqrt{(m)} - \sqrt{(n)}\}^2,$

or $\quad \gamma = \tfrac{1}{2}[1 + \sqrt{(mn)} \pm \sqrt{\{(1 + m)(1 + n)\}}],$

where it is to be noticed that

$$\gamma = \tfrac{1}{2}[1 + \sqrt{(mn)} + \sqrt{\{(1 + m)(1 + n)\}}]$$

is a positive quantity greater than $\sqrt{(mn)}$, say $\gamma = p$,

$$\gamma = \tfrac{1}{2}[1 + \sqrt{(mn)} - \sqrt{\{(1 + m)(1 + n)\}}]$$

is a negative quantity, say $\gamma = -q$, q being positive.

And the order of the lines is as shown in fig. 16.

$\gamma = -\infty$ to $\gamma = -q$, curve is ellipse; $\gamma = -q$, parabola P_2,

$\gamma = -q$ to p, curve is hyperbola; $\gamma = p$, parabola P_1,

$\gamma = p$ to $\gamma = \infty$, ellipse.

Resuming the equation

$$(x-my)(x-ny)+2\{x+y\sqrt{(mn)}\}\gamma z+\gamma^2z^2=0,$$

the coefficients are

$$(a, b, c, f, g, h)=\{1, mn, \gamma^2, \gamma\sqrt{(mn)}, \gamma, -\tfrac{1}{2}(m+n)\},$$

and thence the inverse coefficients are

$$(A, B, C, F, G, H)=[0, 0, -\tfrac{1}{4}(m-n)^2, -\tfrac{1}{2}\gamma\{\sqrt{(m)}+\sqrt{(n)}\}^2,$$
$$-\tfrac{1}{2}\gamma\sqrt{(mn)}\{\sqrt{(m)}+\sqrt{(n)}\}^2, \tfrac{1}{2}\gamma^2\{\sqrt{(m)}+\sqrt{(n)}\}^2],$$

$$K=-\tfrac{1}{4}\gamma^2\{\sqrt{(m)}+\sqrt{(n)}\}^4,$$

or, omitting a factor, the inverse coefficients are

$$(A, B, C, F, G, H)=\left[0, 0, \frac{1}{2\gamma}\{\sqrt{(m)}-\sqrt{(n)}\}^2, 1, \sqrt{(mn)}, -\gamma\right].$$

Considering the line $\lambda x+\mu y+\nu z=0$,

the coordinates of the pole of this line are

$$\begin{aligned} x:y:z= & \qquad\qquad -\gamma\mu+\sqrt{(mn)}\,\nu \\ : & \quad -\gamma\lambda \qquad + \qquad \nu \\ : & \sqrt{(mn)}\,\lambda+\mu+\frac{1}{2\gamma}\{\sqrt{(m)}-\sqrt{(n)}\}^2\nu, \end{aligned}$$

or (what is the same thing) introducing the arbitrary coefficient k, we have

$$kx+\gamma\mu-\nu\sqrt{(mn)}=0,$$
$$ky+\gamma\lambda-\nu\qquad=0,$$
$$kz=\lambda\sqrt{(mn)}+\mu+\frac{1}{2\gamma}\{\sqrt{(m)}-\sqrt{(n)}\}^2\nu;$$

the first two equations give

$$k:\gamma:-1=\nu\{\mu-\lambda\sqrt{(mn)}\}:\nu\{y\sqrt{(mn)}-x\}:\lambda x-\mu y,$$

that is $k=\dfrac{-\nu\{\mu-\lambda\sqrt{(mn)}\}}{\lambda x-\mu y}$, $\gamma=\dfrac{-\nu\{y\sqrt{(mn)}-x\}}{\lambda x-\mu y}$,

or substituting this value of γ in the third equation

$$\frac{\nu\{\mu-\lambda\sqrt{(mn)}\}z}{\lambda x-\mu y}+\{\mu+\lambda\sqrt{(mn)}\}+\frac{\lambda x-\mu y}{x-y\sqrt{(mn)}}\,\frac{\{\sqrt{(m)}-\sqrt{(n)}\}^2}{2}=0,$$

that is

$$(\lambda x-\mu y)^2.\tfrac{1}{2}\{\sqrt{(m)}-\sqrt{(n)}\}^2+\{x-y\sqrt{(mn)}\}(\lambda x-\mu y)\{\mu+\lambda\sqrt{(mn)}\}$$
$$+z\{x-y\sqrt{(mn)}\}\,\nu\{\mu-\lambda\sqrt{(mn)}\}=0,$$

which is the equation of the curve, the locus of the pole of the line $\lambda x + \mu y + \nu z = 0$ in regard to the conic

$$(x - my)(x - ny) + 2\{x + y\sqrt{(mn)}\}\gamma z + \gamma^2 z^2 = 0.$$

In particular, if $\lambda = \mu = \nu = 1$, then for the coordinates of the centre of the conic, we have

$$x : y : z = -\gamma + \sqrt{(mn)} : -\gamma + 1 : \sqrt{(mn)} + 1 + \frac{1}{2\gamma}\{\sqrt{(m)} - \sqrt{(n)}\}^2;$$

and for the locus of the centre,

$$(x-y)^2 . \tfrac{1}{2}\{\sqrt{(m)} - \sqrt{(n)}\}^2 + (x-y)\{x - y\sqrt{(mn)}\}\{1 + \sqrt{(mn)}\}$$
$$+ z\{x - y\sqrt{(mn)}\}\{1 - \sqrt{(mn)}\} = 0,$$

so that the locus is a conic, and it is obvious that this conic is a hyperbola. Putting for greater simplicity

$$\begin{aligned} x - y &= X, \\ x - y\sqrt{(mn)} &= Y, \\ z &= Z, \end{aligned}$$

the equation of the curve of centres is

$$X^2 . \tfrac{1}{2}\{\sqrt{(m)} - \sqrt{(n)}\}^2 + XY\{1 + \sqrt{(mn)}\} + YZ\{1 - \sqrt{(mn)}\} = 0,$$

or, writing this under the form

$$Y[X\{1 + \sqrt{(mn)}\} + Z\{1 - \sqrt{(mn)}\}] + \tfrac{1}{2}\{\sqrt{(m)} - \sqrt{(n)}\}^2 X^2 = 0,$$

the equation is

$$YQ + X^2 = 0,$$

where

$$X = x - y,$$
$$Y = x - y\sqrt{(mn)},$$
$$Q = \frac{2}{\{\sqrt{(m)} - \sqrt{(n)}\}^2}[\{1 + \sqrt{(mn)}\}(x-y) + \{1 - \sqrt{(mn)}\}z]:$$

these values give

$$\begin{aligned} x - y &= X, \\ x - y\sqrt{(mn)} &= Y, \\ \{1 - \sqrt{(mn)}\}z &= \{\sqrt{(m)} - \sqrt{(n)}\}^2 Q + 2\{1 + \sqrt{(mn)}\}X, \end{aligned}$$

or, what is the same thing,

$$\begin{aligned} \{1 - \sqrt{(mn)}\}x &= -\sqrt{(mn)}X + Y, \\ \{1 - \sqrt{(mn)}\}y &= -X + Y, \\ \{1 - \sqrt{(mn)}\}z &= 2\{1 + \sqrt{(mn)}\}X + \{\sqrt{(m)} - \sqrt{(n)}\}^2 Q, \end{aligned}$$

whence also

$$\{1-\surd(mn)\}(x+y+z)=\{1+\surd(mn)\}X+2Y+\{\surd(m)-\surd(n)\}^2Q,$$

or the equation of the line infinity is

$$\{1+\surd(mn)\}X+2Y+\{\surd(m)-\surd(n)\}^2Q=0,$$

a formula which may be applied to finding the asymptotes and thence the centre of the conic

$$YQ+X^2=0.$$

In fact we have identically

$$\{2kx+2ky-(2k+1)z\}^2-(1+4k)(2kx-z)^2$$
$$=4k^2(x+y+z)^2-4k(1+4k)(kx^2+yz),$$

that is, $-4k(1+4k)(kx^2+yz)$

$$=\{2kx+2ky-(2k+1)z\}^2-(1+4k)(2kx-z)^2-4k^2(x+y+z)^2,$$

which, if $x+y+z=0$ is the equation of the line infinity, puts in evidence the asymptotes of the conic $kx^2+yz=0$. Hence writing αx, βy, γz in the place of x, y, z respectively, and $\dfrac{k\alpha^2}{\beta\gamma}=k'$, that is, $k=\dfrac{\beta\gamma}{\alpha^2}k'$, we have

$$-\frac{4\beta\gamma}{\alpha^2}k'\left(1+\frac{4\beta\gamma}{\alpha^2}k'\right)\beta\gamma(k'x^2+yz)$$
$$=\left\{\frac{2\beta\gamma}{\alpha}k'x+\frac{2\beta^2\gamma}{\alpha^2}k'y-\left(2\frac{\beta\gamma}{\alpha^2}k'+1\right)\gamma z\right\}^2$$
$$-\left(1+\frac{4\beta\gamma}{\alpha^2}k'\right)\left(\frac{2\beta\gamma}{\alpha}k'x-\gamma z\right)^2-4\frac{\beta^2\gamma^2}{\alpha^4}k'^2(\alpha x+\beta y+\gamma z)^2,$$

that is, $-4\beta^2\gamma^2k'(\alpha^2+4\beta\gamma k')(k'x^2+yz)$

$$=\{2\alpha\beta\gamma k'x+2\beta^2\gamma k'y-(2\beta\gamma k'+\alpha^2)\gamma z\}^2$$
$$-(\alpha^2+4\beta\gamma k')(2\beta\gamma k'x-\alpha\gamma z)^2-4\beta^2\gamma^2k'^2(\alpha x+\beta y+\gamma z)^2,$$

or, what is the same thing,

$$-4\beta^2k'(\alpha^2+4\beta\gamma k')(k'x^2+yz)=\{2\alpha\beta k'x+2\beta^2k'y-(2\beta\gamma k'+\alpha^2)z\}^2$$
$$-(\alpha^2+4\beta\gamma k')(2\beta k'x-\alpha z)^2-4\beta^2k'^2(\alpha x+\beta y+\gamma z)^2,$$

which, when $\alpha x+\beta y+\gamma z=0$ is the equation of the line infinity, puts in evidence the asymptotes of the conic $k'x^2+yz=0$.

Now writing X, Y, Q in the place of x, y, z; $k'=1$, and $\alpha=\{1+\sqrt{(mn)}\}$, $\beta=2$, $\gamma=\{\sqrt{(m)}-\sqrt{(n)}\}^2$, we have

$$-16\,[\{1+\sqrt{(mn)}\}^2+8\,\{\sqrt{(m)}-\sqrt{(n)}\}^2]\,(YQ+X^2)$$
$$=[4\,\{1+\sqrt{(mn)}\}\,X+8\,Y-(4\,\{\sqrt{(m)}-\sqrt{(n)}\}^2+\{1+\sqrt{(mn)}\}^2)\,Q]^2$$
$$-[\{1+\sqrt{(mn)}\}^2+8\,\{\sqrt{(m)}-\sqrt{(n)}\}^2]\,[4X-\{1+\sqrt{(mn)}\}\,Q]^2$$
$$-16\,[\{1+\sqrt{(mn)}\}\,X+2\,Y+\{\sqrt{(m)}-\sqrt{(n)}\}^2\,Q]^2,$$

and the asymptotes are

$$4\,\{1+\sqrt{(mn)}\}\,X+8\,Y-[4\,\{\sqrt{(m)}-\sqrt{(n)}\}^2+\{1+\sqrt{(mn)}\}^2]\,Q$$
$$=\pm\sqrt{\{1+\sqrt{(mn)}\}^2+8\,\{\sqrt{(m)}-\sqrt{(n)}\}^2}\,[4X-\{1+\sqrt{(mn)}\}\,Q].$$

At the centre

$$4\{1+\sqrt{(mn)}\}\,X+8\,Y-[4\,\{\sqrt{(m)}-\sqrt{(n)}\}^2+\{1+\sqrt{(mn)}\}^2]\,Q=0,$$
$$4X-\{1+\sqrt{(mn)}\}\,Q=0.$$

But the first equation is

$$\{1+\sqrt{(mn)}\}\,[4X-Q\{1+\sqrt{(mn)}\}]+8\,Y-4\,\{\sqrt{(m)}-\sqrt{(n)}\}^2\,Q=0,$$

so that we have

$$4X=\{1+\sqrt{(mn)}\}\,Q,$$
$$2\,Y=\{\sqrt{(m)}-\sqrt{(n)}\}^2\,Q,$$

the first of these is

$$2\,\{\sqrt{(m)}-\sqrt{(n)}\}^2\,(x-y)=\{1+\sqrt{(mn)}\}^2\,(x-y)+(1-mn)\,z,$$

and the two together give

$$2X\,\{\sqrt{(m)}-\sqrt{(n)}\}^2-\{1+\sqrt{(mn)}\}\,Y=0,$$

so that we have

$$2\,\{\sqrt{(m)}-\sqrt{(n)}\}^2\,(x-y)-\{1+\sqrt{(mn)}\}\,\{x-y\,\sqrt{(mn)}\}=0,$$
$$[\{1+\sqrt{(mn)}\}^2-2\,\{\sqrt{(m)}-\sqrt{(n)}\}^2]\,(x-y)+(1-mn)\,z=0,$$

to determine the coordinates of the centre.

The equation of the chord of contact is

$$x+y\,\sqrt{(mn)}+\gamma z=0,$$

which for $\gamma=1$ is parallel to $y=0$ and for $\gamma=\sqrt{(mn)}$ is parallel to $x=0$. But the coordinates of the centre are

$$x:y:z=-\gamma+\sqrt{(mn)}:-\gamma+1:\sqrt{(mn)}+1+\frac{1}{2\gamma}\,\{\sqrt{(m)}-\sqrt{(n)}\}^2,$$

which for $\gamma=1$ give

$$y=0,\ x:z=-1+\sqrt{(mn)}:\sqrt{(mn)}+1+\tfrac{1}{2}\,\{\sqrt{(m)}-\sqrt{(n)}\}^2$$
$$=-2+2\,\sqrt{(mn)}:2+m+n,$$

and for $\gamma = \sqrt{(mn)}$ give

$$x = 0,\ y : z = 1 - \sqrt{(mn)} : \sqrt{(mn)} + 1 + \frac{1}{2\sqrt{(mn)}}\{\sqrt{(m)} - \sqrt{(n)}\}^2$$

$$= 2 - 2\sqrt{(mn)} : 2\sqrt{(mn)} + \frac{m+n}{\sqrt{(mn)}}.$$

The line drawn from the fixed point on the chord of contact to the centre has for its equation

$$x + y\sqrt{(mn)} + \gamma' z = 0,$$

where, writing for x, y, z the coordinates of the centre, we have

$$-\gamma\{1 + \sqrt{(mn)}\} + 2\sqrt{(mn)} + \gamma'\left[\sqrt{(mn)} + 1 + \frac{1}{2\gamma}\{\sqrt{(m)} - \sqrt{(n)}\}^2\right] = 0,$$

that is
$$\gamma' = \frac{\gamma\{1 + \sqrt{(mn)}\} - 2\sqrt{(mn)}}{1 + \sqrt{(mn)} + \frac{1}{2\gamma}\{\sqrt{(m)} - \sqrt{(n)}\}^2},$$

or, what is the same thing,

$$\gamma' - \gamma = \frac{-\gamma\{\sqrt{(m)} + \sqrt{(n)}\}^2}{\{\sqrt{(m)} - \sqrt{(n)}\}^2 + 2\gamma\{1 + \sqrt{(mn)}\}},$$

and consequently $\gamma' = \gamma$ only for $\gamma = 0$.

It is now easy to trace the corresponding positions of the chord of contact through the fixed point $\{x + y\sqrt{(mn)} = 0,\ z = 0\}$, and of the centre on the hyperbola which is the curve of centres. (See fig. 17).

The lines OP_2, OL, $O\Theta$, OP_1, OX, OG, OH are positions of the chord of contact, and the points P_2, L, Θ, P_1, X, G, H, on the hyperbola which is the curve of centres are the corresponding positions of the centre.

Chord of Contact.	*Centre.*
OP_2.	P_2, at infinity on hyperbola.
OL $(z = 0)$.	L, $(z = 0,\ x - y = 0)$.
$O\Theta$.	Θ, the line joining this with O being always behind $O\Theta$.
OP_1.	P_1, at infinity on hyperbola.
OX $\{x + y\sqrt{(mn)} = 0\}$.	X $(x = 0,\ y = 0)$.
OG (parallel to $y = 0$).	G (on line $y = 0$).
OH (parallel to $x = 0$) and so back to	H (on line $x = 0$) and so on to
OP_2.	P_2.

I have treated separately the case $\sqrt{(mn)} = 1$.

Consider the conics which touch the lines $y - x = 0$, $y + x = 0$ and pass through the points

$$\{x = 1,\ y = \sqrt{(1 - c^2)}\},\quad \{x = 1,\ y = -\sqrt{(1 - c^2)}\}.$$

The equation is of the form

$$y^2 - x^2 + k(x - \alpha)^2 = 0,$$

and to determine k, we have

$$1 - c^2 - 1 + k(1 - \alpha)^2 = 0, \text{ and therefore } k = \frac{c^2}{(1 - \alpha)^2}.$$

The equation thus becomes

$$(1 - \alpha)^2 (y^2 - x^2) + c^2 (x - \alpha)^2 = 0,$$

that is $(1 - \alpha)^2 y^2 + \{c^2 - (1 - \alpha)^2\} x^2 - 2c^2 \alpha x + c^2 \alpha^2 = 0,$

or as this may be written

$$(1 - \alpha)^2 y^2 + \{c^2 - (1 - \alpha)^2\} \left\{x - \frac{c^2 \alpha}{c^2 - (1 - \alpha)^2}\right\}^2 - \frac{c^2 \alpha^2 (1 - \alpha)^2}{c^2 - (1 - \alpha)^2} = 0.$$

Hence the nature of the conic depends on the sign of $c^2 - (1 - \alpha)^2$, viz. if this be positive, or α between the limits $1 + c$, $1 - c$, the curve is an ellipse,

$$x\text{-coordinate of centre} = \frac{c^2 \alpha}{c^2 - (1 - \alpha)^2},$$

which is positive,

$$\left.\begin{aligned} x\text{-semi-axis} &= \frac{\pm c\alpha (1 - \alpha)}{c^2 - (1 - \alpha)^2} \\ y\text{-semi-axis} &= \frac{c\alpha}{\sqrt{\{c^2 - (1 - \alpha)^2\}}} \end{aligned}\right\}.$$

The coordinate of centre for $\alpha = 1 + c$ is $= +\infty$ (the curve being in this case a parabola P_1) and for $\alpha = 1 - c$ it is also $= +\infty$ (the curve being in this case a parabola P_2). The coordinate has a minimum value corresponding to $\alpha = \sqrt{(1 - c^2)}$, viz. this is $= \frac{1}{2}\{1 + \sqrt{(1 - c^2)}\}$.

Hence as (α) passes from $1 + c$ to $\sqrt{(1 - c^2)}$, the coordinate of the centre passes from ∞ to its minimum value $\frac{1}{2}\{1 + \sqrt{(1 - c^2)}\}$; in the passage we have $\alpha = 1$ giving the coordinate $= 1$, the conic being in this case a pair of coincident lines $(x - 1)^2 = 0$. And as (α) passes from the foregoing value $\sqrt{(1 - c^2)}$ to $1 - c$, the coordinate of the centre passes from the minimum value $\frac{1}{2}\{1 + \sqrt{(1 - c^2)}\}$ to ∞.

The curve is a hyperbola if α lies without the limits $1+c$, $1-c$,

$$x\text{-coordinate of centre} \quad = \frac{-c^2\alpha}{(1-\alpha)^2-c^2},$$

which is of sign $-\alpha$,

$$x\text{-semi-axis} \quad = \frac{\pm c\alpha(1-\alpha)}{(1-\alpha)^2-c^2},$$

$$y\text{-semi-axis} \quad = \frac{\pm c\alpha}{\sqrt{\{(1-\alpha)^2-c^2\}}},$$

$$\text{semi-aperture of asymptotes} = \tan^{-1}\sqrt{\left\{1-\frac{c^2}{(1-\alpha)^2}\right\}},$$

which for $\alpha=1\pm c$ is $=0$ (parabola), but increases as $1-\alpha$ increases positively or negatively, becoming $=45°$ for $\alpha=\pm\infty$ (the asymptotes being in this case the pair of lines $y^2-x^2=0$):

$$\alpha=+\infty,\ \text{coordinate of centre is} = \quad 0,$$

$$\alpha=1+c,\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots = -\infty,$$

so that α diminishing from ∞ to $1+c$, the coordinate of the centre moves constantly in the same direction from 0 to $-\infty$,

$$\alpha=1-c,\ \text{coordinate of centre is} = -\infty,$$

$$\alpha=0,\quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots = \quad 0,$$

the hyperbola being in this case the pair of lines $y^2=(1-c^2)x^2$.

α negative, the coordinate of centre becomes positive, viz. as α passes from $\alpha=0$ to $\alpha=-\sqrt{(1-c^2)}$, the coordinate of centre passes from 0 to a maximum positive value $\frac{1}{2}\{1-\sqrt{(1-c^2)}\}$, and then as α passes from $-\sqrt{(1-c^2)}$ to $-\infty$, the coordinate of centre diminishes from $\frac{1}{2}\{1-\sqrt{(1-c^2)}\}$ to 0. It is to be remarked that α being negative, the lines $y^2-x^2=0$ are touched by the branch on the negative side of the origin, that is the branch not passing through the two points $x=1$, $y=\pm\sqrt{(1-c^2)}$.

ON THE CONICS WHICH TOUCH THREE GIVEN LINES AND PASS THROUGH A GIVEN POINT.

By Professor CAYLEY.

CONSIDER the triangles which touch three given lines; the three lines form a triangle, and the lines joining the angles of the triangle with the points of contact of the opposite sides respectively meet in a point S: conversely given the three lines and the point S, then joining this point with the angles of the triangle the joining lines meet the opposite sides respectively in three points which are the points of contact with the three given lines respectively of a conic; such conic is determinate and unique. Suppose now that the conic passes through a given point; the point S is no longer arbitrary, but it must lie on a certain curve; and this curve being known, then taking upon it any point whatever for the point S, and constructing as before the conic which corresponds to such point, the conic in question will pass through the given point, and will thus be a conic touching the three given lines and passing through the given point. And the series of such conics corresponds of course to the series of points on the curve.

I proceed to find the curve which is the locus of the point S.

We may take $x=0$, $y=0$, $z=0$ for the equations of the given lines, and $x:y:z=1:1:1$ for the coordinates of the given point. The equation of a conic touching the three given lines is

$$a\sqrt{(x)}+b\sqrt{(y)}+c\sqrt{(z)}=0,$$

and the coordinates of the corresponding point S are as $\frac{1}{a^2}:\frac{1}{b^2}:\frac{1}{c^2}$, that is, taking (x, y, z) for the coordinates of the point in question, we have

$$a:b:c=\frac{1}{\sqrt{(x)}}:\frac{1}{\sqrt{(y)}}:\frac{1}{\sqrt{(z)}},$$

the condition in order that the conic may pass through the given point is $a+b+c=0$, and we thus find for the curve, which is the locus of the point S, the equation

$$\frac{1}{\sqrt{(x)}}+\frac{1}{\sqrt{(y)}}+\frac{1}{\sqrt{(z)}}=0,$$

or, what is the same thing,

$$\sqrt{(yz)} + \sqrt{(zx)} + \sqrt{(xy)} = 0,$$

the rationalised form of which is

$$y^2z^2 + z^2x^2 + x^2y^2 - 2xyz(x + y + z) = 0.$$

This is a quartic curve with three cusps, viz. each angle of the triangle is a cusp; and by considering for example the cusp ($y = 0$, $z = 0$) and writing the equation under the form

$$x^2(y - z)^2 - 2x(yz^2 + y^2z) + y^2z^2 = 0,$$

we see that the tangent at the cusp in question is the line $y - z = 0$; that is, the tangents at the three cusps are the lines joining these points respectively with the given point (1, 1, 1). Each cuspidal tangent meets the curve in the cusp counting as three points and in a fourth point of intersection, the coordinates whereof in the case of the tangent $y - z = 0$, are at once found to be $x : y : z = 1 : 4 : 4$, or say this is the point (1, 4, 4); the point on the tangent $z - x = 0$ is of course (4, 1, 4), and that on the tangent $x - y = 0$ is (4, 4, 1). To find the tangents at these points respectively, I remark that the general equation of the tangent is

$$(X\delta_x + Y\delta_y + Z\delta_z)\left\{\frac{1}{\sqrt{(x)}} + \frac{1}{\sqrt{(y)}} + \frac{1}{\sqrt{(z)}}\right\} = 0,$$

that is

$$\frac{X}{x^{\frac{3}{2}}} + \frac{Y}{y^{\frac{3}{2}}} + \frac{Z}{z^{\frac{3}{2}}} = 0,$$

or for the point (1, 4, 4) the equation of the tangent is $8X + Y + Z = 0$, or say $8x + y + z = 0$; that is, the tangent passes through the point $x = 0$, $x + y + z = 0$, being the point of intersection of the line $x = 0$ with the line $x + y + z = 0$, which is the harmonic of the given point (1, 1, 1) in regard to the triangle; the tangents at the points (1, 4, 4), (4, 1, 4), (4, 4, 1) respectively pass through the points of intersection of the harmonic line $x + y + z = 0$ with the three given lines respectively.

In the case where the given point lies within the triangle, the curve the locus of S lies wholly within the triangle, and is of the form shown in fig. 18; it is clear that in this case the conics of the system are all of them ellipses; there are however three limiting forms, viz. the line joining the given point with any angle of the triangle, such line being regarded as a twofold line or pair of coincident lines, is a conic of

the system. The discussion of the two cases in which the given point lies outside the triangle, viz. in the infinite space bounded by two sides produced, or in the infinite space bounded by a side and two sides produced, may be effected without much difficulty.

ON CERTAIN TRANSFORMATIONS IN THE CALCULUS OF OPERATIONS.

By WILLIAM WALTON, M.A., Trinity College, Cambridge.

1. I PROPOSE in the first place to prove the following theorem, viz. that, for all positive integral values of n,

$$\tfrac{1}{2}\,\frac{\dfrac{d}{d0}}{1-e^{\frac{d}{d0}}}\{(1+20)^{2n+1}+(1-20)^{2n+1}\}=\frac{\dfrac{d}{d0}}{1-e^{\frac{d}{d0}}}\{(1+0)^{2n+1}+(1-0)^{2n+1}\}.$$

Let N denote the result of the subtraction of the latter from the former expression: then, replacing $e^{\frac{d}{d0}}$ by $1+\Delta$, we have

$$N=\frac{\log(1+\Delta)}{\Delta}\{(1+0)^{2n+1}+(1-0)^{2n+1}-\tfrac{1}{2}(1+20)^{2n+1}-\tfrac{1}{2}(1-20)^{2n+1}\}.$$

Now, r being any positive integer, $(1+\Delta)^{2r}$ operating on the expression

$$(1+0)^{2n+1}+(1-0)^{2n+1},$$

produces the same result as $(1+\Delta)^r$ operating on the expression

$$(1+20)^{2n+1}+(1-20)^{2n+1}.$$

Hence, replacing, in the last two elements of the expression for N, $\Delta+1$ by $(\Delta+1)^2$, $(1+20)^{2n+1}$ by $(1+0)^{2n+1}$, and $(1-20)^{2n+1}$ by $(1-0)^{2n+1}$, we get

$$N=\log(1+\Delta)\left\{\frac{1}{\Delta}-\frac{1}{(\Delta+1)^2-1}\right\}\{(1+0)^{2n+1}+(1-0)^{2n+1}\}.$$

Again,

$$(1+0)^{2n+1}=(1+\Delta)\,0^{2n+1},\quad (1+\Delta)(1-0)^{2n+1}=-\,0^{2n+1},$$

and consequently

$$(1+0)^{2n+1}+(1-0)^{2n+1}=\left(1+\Delta-\frac{1}{1+\Delta}\right)0^{2n+1}.$$

Hence $N=\log(1+\Delta)\left\{\frac{1}{\Delta}-\frac{1}{(\Delta+1)^2-1}\right\}\left(1+\Delta-\frac{1}{1+\Delta}\right)0^{2n+1}$

$$=\log(1+\Delta)\,0^{2n+1}$$

$$=\frac{d}{d0}\,0^{2n+1}=(2n+1)\,0^{2n}=0.$$

This result establishes our theorem.

2. I proceed now to apply the preceding theorem to the demonstration of a certain equation connecting together the first n of Bernoulli's numbers.

By the preceding theorem we get

$$0=\frac{1}{2}\,\frac{\frac{d}{d0}}{1-e^{\frac{d}{d0}}}\{(1+20)^{2n+1}+(1-20)^{2n+1}\}-\frac{\frac{d}{d0}}{1-e^{\frac{d}{d0}}}\{(1+0)^{2n+1}+(1-0)^{2n+1}\}$$

$$=1-\frac{1}{2}\left\{\frac{\frac{d}{d0}}{e^{\frac{d}{d0}}-1}-1\right\}\{(1+20)^{2n+1}+(1-20)^{2n+1}\}$$

$$+\left\{\frac{\frac{d}{d0}}{e^{\frac{d}{d0}}-1}-1\right\}\{(1+0)^{2n+1}+(1-0)^{2n+1}\}:$$

and therefore, denoting $\dfrac{\frac{d}{d0}}{e^{\frac{d}{d0}}-1}-1$ by E,

$$0=\frac{1}{2}\,\frac{1}{1.2.3\ldots(2n+1)}$$

$$-\tfrac{1}{4}E\{(1+20)^{2n+1}+(1-20)^{2n+1}\}\,\frac{1}{1.2.3\ldots(2n+1)}$$

$$+\tfrac{1}{2}E\{(1+0)^{2n+1}+(1-0)^{2n+1}\}\,\frac{1}{1.2.3\ldots(2n+1)}$$

$$= \tfrac{1}{2}\,\frac{1}{\lfloor\!\underline{2n+1}}$$

$$-\tfrac{1}{2}E\left\{\frac{(20)^2}{1.2\times\lfloor\!\underline{2n-1}}+\frac{(20)^4}{\lfloor\!\underline{4}\times\lfloor\!\underline{2n-3}}+\frac{(20)^6}{\lfloor\!\underline{6}\times\lfloor\!\underline{2n-5}}+\ldots+\frac{(20)^{2n}}{\lfloor\!\underline{2n}\times 1}\right\}$$

$$+\ E\left\{\frac{0^2}{1.2\times\lfloor\!\underline{2n-1}}+\frac{0^4}{\lfloor\!\underline{4}\times\lfloor\!\underline{2n-3}}+\frac{0^6}{\lfloor\!\underline{6}\times\lfloor\!\underline{2n-5}}+\ldots+\frac{0^{2n}}{\lfloor\!\underline{2n}\times 1}\right\}$$

$$= \tfrac{1}{2}\,\frac{1}{\lfloor\!\underline{2n+1}}$$

$$-E\left\{\frac{0^2}{1.2}\,\frac{2^1-1}{\lfloor\!\underline{2n-1}}+\frac{0^4}{\lfloor\!\underline{4}}\cdot\frac{2^3-1}{\lfloor\!\underline{2n-3}}+\frac{0^6}{\lfloor\!\underline{6}}\cdot\frac{2^5-1}{\lfloor\!\underline{2n-5}}+\ldots+\frac{0^{2n}}{\lfloor\!\underline{2n}}\cdot\frac{2^{2n-1}-1}{1}\right\}$$

$$= \tfrac{1}{2}\,\frac{1}{\lfloor\!\underline{2n+1}}$$

$$-\frac{2^1-1}{1.2}\cdot\frac{B_1}{\lfloor\!\underline{2n-1}}+\frac{2^3-1}{\lfloor\!\underline{4}}\cdot\frac{B_3}{\lfloor\!\underline{2n-3}}-\ldots+(-1)^n\,\frac{2^{2n-1}-1}{\lfloor\!\underline{2n}}\cdot\frac{B_{2n-1}}{1},$$

an equation connecting the first n of Bernoulli's numbers.

3. It will be convenient, in relation to some of the following investigations, to obtain the sum of the infinite series

$$\sin\theta+\tfrac{1}{2}\sin 2\theta+\tfrac{1}{3}\sin 3\theta+\ldots.$$

Let u represent the series: then, C being an arbitrary constant,

$$2u=\left\{\left(e^{\theta\frac{d}{d0}}-e^{-\theta\frac{d}{d0}}\right)+\tfrac{1}{2}\left(e^{2\theta\frac{d}{d0}}-e^{-2\theta\frac{d}{d0}}\right)+\tfrac{1}{3}\left(e^{3\theta\frac{d}{d0}}-e^{-3\theta\frac{d}{d0}}\right)+\ldots\right\}(\sin 0+C)$$

$$=\log\left(\frac{1-e^{-\theta\frac{d}{d0}}}{1-e^{\theta\frac{d}{d0}}}\right)(C+\sin 0)$$

$$=\log\left\{(-1)\,e^{-\theta\frac{d}{d0}}\right\}(C+\sin 0)$$

$$=\left\{(2\lambda+1)\,\pi\sqrt{(-1)}-\theta\frac{d}{d0}\right\}(C+\sin 0)$$

$$=(2\lambda+1)\,\pi C\sqrt{(-1)}-\theta.$$

Now $\dfrac{du}{d\theta}=\cos\theta+\cos 2\theta+\cos 3\theta+\ldots+\cos n\theta,\ n=\infty,$

$$=\frac{\sin\dfrac{n\theta}{2}\,.\cos\dfrac{n+1}{2}\,\theta}{\sin\dfrac{\theta}{2}},\ n=\infty.$$

Now, while θ lies between the limits 0 and 2π, $\dfrac{du}{d\theta}$ is finite: hence, within these limits, u is not discontinuous; put $\theta=\pi$: then

$$0=(2\lambda+1)\,\pi C\sqrt{(-1)}-\pi;$$

and therefore, θ being confined within the said limits,

$$\sin\theta+\tfrac{1}{2}\sin 2\theta+\tfrac{1}{3}\sin 3\theta+\ldots=\frac{\pi-\theta}{2}.$$

Cor. If θ lies between $-\pi$ and $+\pi$, we may write $\pi-\theta$ for θ in the preceding equation: thus we get

$$\sin\theta-\tfrac{1}{2}\sin 2\theta+\tfrac{1}{3}\sin 3\theta-\ldots=\frac{\theta}{2}.$$

4. I proceed next to the transformation of the infinite series

$$f(\theta)=\frac{\sin\theta}{1^{2n+1}}-\frac{\sin 2\theta}{2^{2n+1}}+\frac{\sin 3\theta}{3^{2n+1}}-\ldots$$

into a series ascending by powers of θ.

Since there can evidently be only odd powers of θ in the result, we have, by McLaurin's Theorem,

$$f(\theta)=\theta f'(0)+\frac{\theta^3}{1.2.3}f'''(0)+\frac{\theta^5}{1.2.3.4.5}f^{v}(0)+\ldots.$$

Now, B_1, B_3, B_5,...denoting Bernoulli's Numbers,

$$f'(0)=\frac{1}{1^{2n}}-\frac{1}{2^{2n}}+\frac{1}{3^{2n}}-\ldots$$

$$=\frac{2^{2n-1}-1}{\lfloor 2n}\pi^{2n}B_{2n-1},$$

by a well known theorem:

$$f'''(0)=-\left(\frac{1}{1^{2n-2}}-\frac{1}{2^{2n-2}}+\frac{1}{3^{2n-2}}-\ldots\right)$$

$$=-\frac{2^{2n-3}-1}{\lfloor 2n-2}\pi^{2n-2}B_{2n-3},$$

....

....

$$f^{2n-1}(0) = (-1)^{n-1}\left(\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \dots\right)$$

$$= (-1)^{n-1}\,\frac{2^1-1}{1.2}\,\pi^2 B_1.$$

Also $\quad f^{2n}(\theta) = (-1)^n\left(\sin\theta - \tfrac{1}{2}\sin 2\theta + \tfrac{1}{3}\sin 3\theta - \dots\right)$

$$= (-1)^n\,\frac{\theta}{2},$$

if θ lie between $-\pi$ and $+\pi$: hence

$$f^{2n+1}(0) = (-1)^n.\tfrac{1}{2}.$$

Hence, provided that θ lies between $-\pi$ and $+\pi$,

$$\begin{aligned} f(\theta) = {} & \frac{2^{2n-1}-1}{\lfloor\underline{2n}}\,\pi^{2n} B_{2n-1}\,\theta \\ & - \frac{2^{2n-3}-1}{\lfloor\underline{2n-2}}\,\pi^{2n-2} B_{2n-3}\,\frac{\theta^3}{\lfloor\underline{3}} \\ & + \frac{2^{2n-5}-1}{\lfloor\underline{2n-4}}\,\pi^{2n-4} B_{2n-5}\,\frac{\theta^5}{\lfloor\underline{5}} \\ & - \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ & + (-1)^{n-1}\,\frac{2^1-1}{1.2}\,\pi^2 B_1\,\frac{\theta^{2n-1}}{\lfloor\underline{2n-1}} + \tfrac{1}{2}(-1)^n\,\frac{\theta^{2n+1}}{\lfloor\underline{2n+1}}. \end{aligned}$$

This transformation has been deduced by Mr. Blissard as a particular consequence of his Theory of Generic Equations. (*Quarterly Journal*, Vol. IV., p. 296).

5. Let $\quad u_{2n+1} = \dfrac{\sin\theta}{1^{2n+1}} - \dfrac{\sin 2\theta}{2^{2n+1}} + \dfrac{\sin 3\theta}{3^{2n+1}} - \dots,$

and let $\;\psi(\theta) = u_1\phi(\pi) - u_3\dfrac{d^2}{d\pi^2}\phi(\pi) + u_5\dfrac{d^4}{d\pi^4}\phi(\pi) - \dots,$

where ϕ indicates such functionality that $\phi(\theta)$ is expressible in a series involving only odd powers of θ. Then, attending to the expression for u_{2n+1} given in section (4), we see that, θ lying between $-\pi$ and $+\pi$, the term involving θ^{2r+1}, in the expansion of $\psi(\theta)$ by ascending powers of θ, is equal to

$$\frac{\theta^{2r+1}}{\lfloor\underline{2r+1}}\left(\frac{1}{2}\,\frac{d^{2r}}{d\pi^{2r}} - \frac{2^1-1}{1.2}\,\pi^2 B_1\,\frac{d^{2r+2}}{d\pi^{2r+2}} + \frac{2^3-1}{1.2.3.4}\,\pi^4 B_3\,\frac{d^{2r+4}}{d\pi^{2r+4}} - \dots\right)\phi(\pi).$$

The coefficient of $\dfrac{\theta^{2r+1}}{\lfloor 2r+1}$ in this expression is equal to

$$\frac{1}{2}\left\{\pi\phi^{2r+1}(0)+\frac{\pi^3}{1.2.3}\phi^{2r+3}(0)+\frac{\pi^5}{1.2.3.4.5}\phi^{2r+5}(0)+\ldots\right\}$$

$$-\frac{2^1-1}{1.2}B_1\left\{\pi^3\phi^{2r+3}(0)+\frac{\pi^5}{1.2.3}\phi^{2r+5}(0)+\frac{\pi^7}{1.2.3.4.5}\phi^{2r+7}(0)+\ldots\right\}$$

$$+\frac{2^3-1}{1.2.3.4}B_3\left\{\pi^5\phi^{2r+5}(0)+\frac{\pi^7}{1.2.3}\phi^{2r+7}(0)+\frac{\pi^9}{1.2.3.4.5}\phi^{2r+9}(0)+\ldots\right\}$$

$$-\ \&c.$$

$$=\tfrac{1}{2}\pi\phi^{2r+1}(0)+\pi^3\left\{\frac{1}{2}\cdot\frac{1}{1.2.3}-\frac{2^1-1}{1.2}\cdot\frac{B_1}{1}\right\}\phi^{2r+3}(0)$$

$$+\pi^5\left\{\frac{1}{2}\cdot\frac{1}{1.2.3.4.5}-\frac{2^1-1}{1.2}\cdot\frac{B_1}{1.2.3}+\frac{2^3-1}{1.2.3.4}\cdot\frac{B_3}{1}\right\}\phi^{2r+5}(0)$$

$$+\pi^7\left\{\frac{1}{2}\cdot\frac{1}{1.2.3.4.5.6.7}-\frac{2^1-1}{1.2}\cdot\frac{B_1}{1.2.3.4.5}\right.$$

$$\left.+\frac{2^3-1}{1.2.3.4}\cdot\frac{B_3}{1.2.3}-\frac{2^5-1}{1.2.3.4.5.6}\cdot\frac{B_5}{1}\right\}\phi^{2r+7}(0)$$

$$+\ \&c.$$

In this expression, as we see by the equation connecting Bernoulli's Numbers, which I have established in section (2), the coefficients of π^3, π^5, π^7,... are all equal to zero: hence we see that, in the expansion of $\psi(\theta)$ by ascending powers of θ, the term involving θ^{2r+1} is equal to

$$\tfrac{1}{2}\pi\theta^{2r+1}\frac{\phi^{2r+1}(0)}{1.2.3\ldots(2r+1)}:$$

the expression $\psi(\theta)$ is therefore equal to $\frac{1}{2}\pi\phi(\theta)$.

This conclusion constitutes Fourier's celebrated Theorem, which he has given in his *Théorie Analytique de la Chaleur*, p. 230. Fourier's demonstration, it may be remarked, extends over about eighteen small quarto pages.

October 16, 1863.

ON LINEAR DIFFERENTIAL EQUATIONS WITH PARTICULAR INTEGRALS ALL OF THE SAME FORM.

By Dr. Adolph Steen, Professor of Mathematics in the University of Copenhagen.

A paper by the Honourable James Cockle, in Queensland, in Australia, in the February Number of this *Journal*, treats on some linear differential equations of the third order with variable coefficients, which become constant by a change of the independent variable. The author gives the general form of these differential equations thus:

$$\frac{d^3y}{dx^3}+\left(3\frac{d\lambda}{dx}+ae^{-\lambda}\right)\frac{d^2y}{dx^2}$$
$$+\left\{\frac{d^2\lambda}{dx^2}+2\left(\frac{d\lambda}{dx}\right)^2+ce^{-\lambda}\frac{d\lambda}{dx}+be^{-2\lambda}\right\}\frac{dy}{dx}+de^{-3\lambda}y=0,$$

and he asserts that the assumption of

$$t=\int e^{-\lambda}dx$$

will produce an equation with constant coefficients.*

Having just last winter communicated to the Royal Society of Sciences at Copenhagen a more general theory of differential equations, including the case presented by Mr. Cockle as a special one, I take this opportunity to publish the result of my investigations for the readers of this *Journal*.

1. The general linear differential equation of the n^{th} order

$$P\frac{d^ny}{dx^n}+P_1\frac{d^{n-1}y}{dx^{n-1}}+P_2\frac{d^{n-2}y}{dx^{n-2}}+\ldots+P_ny=0\ldots\ldots(1),$$

P_n being constant, $P_1, P_2, \ldots P_{n-1}$ functions of x, is supposed to have n particular integrals all of the same form, viz.

$$y=ce^{\int\frac{dx}{F(m,x)}}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

only differing from each other in the values of m.

* Only there is here a slight mistake, there will be no constant coefficients unless $c=a$.

The complete integral is then

$$y = \Sigma c_r e^{\int \frac{dx}{F(m_r, x)}}.$$

If we know the form (2), its substitution in (1) will necessarily produce an equation to calculate the n values of m, thus

$$\phi(m)\, \psi(m, x) = 0.$$

It should only be observed, that s equal roots will change the corresponding particular integrals into the forms

$$f(m, x),\ \frac{df(m, x)}{dm},\ \frac{d^2 f(m, x)}{dm^2} \cdots \frac{d^{s-1} f(m, x)}{dm^{s-1}}.$$

A general demonstration of this theorem and of some others has been given by me in a paper inserted in the *Mathematisk Tidsskrift*, Copenhagen, 1859.

The differential equation of the first order, without the constant c corresponding to (2), viz.

$$F(m, x)\frac{dy}{dx} = y \ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

exists for all values of m as a differential equation belonging to the primitive (2), but it will only correspond to (1), when m has the n values just mentioned.

Now a new differential equation, of the $(n+1)^{\text{th}}$ order, can be formed in two different ways with all the same particular integrals belonging to (1), and, moreover, y equal a constant. First it can be obtained by differentiation of (1), giving

$$P\frac{d^{n+1}y}{dx^{n+1}} + \left(P_1 + \frac{dP}{dx}\right)\frac{d^n y}{dx^n} + \ldots + P_n \frac{dy}{dx} = 0 \ldots. (4).$$

Next, another differential equation with the same particular integrals is produced by substitution into (1) of the expression (3) for y and its differential coefficients deduced from (3). Thus will be found

$$PF(m, x)\frac{d^{n+1}y}{dx^{n+1}} + \left\{P_1 F(m, x) + nP\frac{dF(m, x)}{dx}\right\}\frac{d^n y}{dx^n} + \ldots$$

$$+ P_n F(m, x)\frac{dy}{dx} = 0 \ldots\ldots\ldots\ldots (5).$$

But two linear differential equations of the form (1) with the same particular integrals can generally only differ by a

factor common to all the terms of one of the equations.* This theorem, however, cannot always be applied to the equations (4) and (5), because the last of them contains m, which may assume n values, and in the meantime m does not appear at all in the first. But when $F(m, x)$ contains m in such a manner, that m in (5) only exists in a common factor of all the terms, the division with that factor produces a new equation without m, which must be identical with (4). Under that condition we shall have

$$nP\frac{dF(m, x)}{dx} = F(m, x)\frac{dP}{dx},$$

whence, the constant being a,

$$F(m, x) = \sqrt[n]{(aP)}.$$

Here m can only be contained in the arbitrary constant a, therefore putting $\sqrt[n]{(a)} = \frac{1}{m}$, we find, for the general form (2),

$$y = ce^{m\int\frac{dx}{\sqrt[n]{(P)}}} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6)$$

satisfying the supposed condition to make m appear only in a common factor of all the terms in (5).

2. Among all the most elementary forms for y, composed of a function X of x, and a quantity m assuming n values, viz.

$$m + X,\ mX,\ X^m,\ m^x,\ \log_m X,\ \log_x m,$$

only one, X^m, is a possible common form for the particular integrals of the equation (1).

* This theorem is very simply demonstrated in the following manner: Let y_p be a particular integral, $y_p^{(q)}$ its differential coefficient of the q^{th} order and

$$\Delta_r = \Sigma y_1^{(n)} y_2^{(n-1)} \ldots y_{n-r}^{(r+1)} y_{n-r+1}^{(r-1)} \ldots y'_{n-1} y_n,$$

the determinant of these quantities, then the two linear differential equations with the general terms

$$P_r\frac{d^{n-r}y}{dx^{n-r}} \text{ and } Q_r\frac{d^{n-r}y}{dx^{n-r}}$$

will have their coefficients determined by

$$\left.\begin{array}{l} \dfrac{P_0}{\Delta_n} = \dfrac{P_1}{\Delta_{n-1}} = \ldots = \dfrac{P_{n-r}}{\Delta_r} = \ldots = \dfrac{P_n}{\Delta_0} \\ \dfrac{Q_0}{\Delta_n} = \dfrac{Q_1}{\Delta_{n-1}} = \ldots = \dfrac{Q_{n-r}}{\Delta_r} = \ldots = \dfrac{Q_n}{\Delta_0} \end{array}\right\},\ \text{whence}\ \frac{P_0}{Q_0} = \frac{P_1}{Q_1} = \ldots = \frac{P_n}{Q_n}. \quad \text{Q. E. D.}$$

Taking first

$$y = \Sigma c_r(m_r + X) = \Sigma c_r m_r + X\Sigma c_r,$$

or

$$y = \Sigma c_r m_r X = X\Sigma c_r m_r,$$

we shall reduce the complete integral to one or two particular integrals instead of n.

Next, we have

$$m^X = e^{Xlm},$$

being of the form X^m.

Lastly, it is easily seen that

$$\log_m X = \frac{lX}{lm},$$

$$\log_x m = \frac{lm}{lX},$$

both of the form mX.

By these remarks is generally demonstrated the following:

THEOREM.

A linear differential equation of the form (1), *with all its particular integrals of the same elementary combination of a function X of x and a quantity m, assuming n values, can only have them of the form*

$$y = e^{m\int\frac{dx}{\sqrt[n]{(P)}}},$$

so that its coefficients become constant, if the independent variable is changed into

$$t = \int\frac{dx}{\sqrt[n]{(P)}}.$$

3. The general form of the differential equations, satisfying the theorem just mentioned, can easily be found by a series of integrations of linear differential equations of the first order determining $P_1, P_2, P_3 \dots P_n$. But the indicated change of the independent variable, or the substitution of the general expression for y will sufficiently show in every case, if the proposed equation is integrable in this way or not.

Here we shall find the general expression for P_1, that we may be able to make a new conclusion to other differential equations with particular integrals of the same form. Comparing the coefficients of $\dfrac{d^{n-1}y}{dx^{n-1}}$ in the equations (4) and (5), we shall find

$$\frac{n(n-1)}{2} P \frac{d^2 P^{\frac{1}{n}}}{dx^2} + \frac{n-1}{1} P_1 \frac{dP^{\frac{1}{n}}}{dx} = P^{\frac{1}{n}} \frac{dP_1}{dx},$$

or $$\frac{dP_1}{dx} - \frac{n-1}{1}\frac{dlP^{\frac{1}{n}}}{dx}P_1 = \frac{n(n-1)}{2}P^{1-\frac{1}{n}}\frac{d^2P^{\frac{1}{n}}}{dx^2},$$

consequently $$P_1 = \frac{n-1}{2}\frac{dP}{dx} + AP^{\frac{n-1}{n}},$$

A being the arbitrary constant.

The equation

$$P\frac{d^nz}{dx} + \left(\frac{n-1}{2}\frac{dP}{dx} + AP^{\frac{n-1}{n}}\right)\frac{d^{n-1}z}{dx^{n-1}} + \ldots + P_nz = 0 \ldots (7),$$

where P_n is a constant, may therefore have particular integrals all of the form (6). Now let (7) be in this case, and let us put

$$z = \frac{y}{X},$$

X being a function of x, so that we obtain a new form

$$P\frac{d^ny}{dx^n} - \left(\frac{nP}{x}\frac{dX}{dx} - \frac{n-1}{2}\frac{dP}{dx} - AP^{\frac{n-1}{n}}\right)\frac{d^{n-1}y}{dx^{n-1}} + \ldots + P_ny = 0 \qquad \ldots\ldots\ldots\ldots\ldots\ldots (8),$$

wherein can appear however more than one term containing y, even with variable coefficients.

Comparing the equations (8) and (1), which in this case may have P_n, consisting of more terms than only one constant, we find for the calculation of X,

$$\frac{nP}{X}\frac{dX}{dx} - \frac{n-1}{2}\frac{dP}{dx} - AP^{\frac{n-1}{n}} = -P_1,$$

consequently $$X = P^{\frac{n-1}{2n}} e^{\frac{A}{n}\int\frac{dx}{\sqrt[n]{(P)}} - \frac{1}{n}\int\frac{P_1}{P}dx}.$$

For this X (8) will have particular integrals all of the form

$$y = Xe^{m\int\frac{dx}{\sqrt[n]{(P)}}} = P^{\frac{n-1}{2n}} e^{m\int\frac{dx}{\sqrt[n]{(P)}} - \frac{1}{n}\int\frac{P_1}{P}dx} \ldots\ldots\ldots (9),$$

after a change of $\frac{A}{n} + m$ into m.

When the substitution of (9) into (1) produces an equation in m alone, the proposed equation will have n particular integrals of the form (9), though with the above mentioned modification in the case of equal roots.

4. A very interesting example of the last kind is

$$x^{2n}\frac{d^n y}{dx^n} - a^n y = 0.$$

Here (9) gives $$y = x^{n-1}e^{\frac{m}{x}},$$

which may be the common form of the particular integrals. In order to try if this is the case there is wanted the general expression for

$$\frac{d^n y}{dx^n} = \frac{d^n x^{n-1}e^{\frac{m}{x}}}{dx^n}.$$

Now it is easily found directly that

$$\frac{dy}{dx} = (n-1)\,x^{n-2}e^{\frac{m}{x}} - mx^{n-3}e^{\frac{m}{x}},$$

$$\frac{d^2y}{dx^2} = (n-1)(n-2)\,x^{n-3}e^{\frac{m}{x}} - 2m(n-2)\,x^{n-4}e^{\frac{m}{x}} + m^2x^{n-5}e^{\frac{m}{x}},$$

$$\frac{d^3y}{dx^3} = (n-1)(n-2)(n-3)\,x^{n-4}e^{\frac{m}{x}} - 3m(n-2)(n-3)\,x^{n-5}e^{\frac{m}{x}} + 3m^2(n-3)\,x^{n-6}e^{\frac{m}{x}} - m^3x^{n-7}e^{\frac{m}{x}},$$

....

and by induction, that

$$\begin{aligned}\frac{d^ry}{dx^r} &= (n-1)(n-2)\ldots(n-r)\,x^{n-r-1}e^{\frac{m}{x}}\\ &- \frac{r}{1}\,m(n-2)\ldots(n-r)\,x^{n-r-2}e^{\frac{m}{x}}\\ &+ \frac{r(r-1)}{1.2}\,m^2(n-3)\ldots(n-r)\,x^{n-r-3}e^{\frac{m}{x}}\ldots\\ &+ (-1)^p\frac{r(r-1)\ldots(r-p+1)}{1.2\ldots p}\,m^p(n-p-1)\ldots(n-r)\,x^{n-r-p-1}e^{\frac{m}{x}}\\ &+ (-1)^{p+1}\frac{r(r-1)\ldots(r-p)}{1.2\ldots p}\,m^{p+1}(n-p-2)\ldots(n-r)\,x^{n-r-p-2}e^{\frac{m}{x}}\ldots\\ &+ (-1)^r m^r x^{n-2r-1}e^{\frac{m}{x}}.\end{aligned}$$

The correctness of this formula results from a differentiation which will give a new formula for $\frac{d^{r+1}y}{dx^{r+1}}$ depending on $r+1$ just the same as $\frac{d^ry}{dx^r}$ depends on r. Here it will be enough

to show what is the general term of this new formula. Differentiating the first of the two general terms in $\frac{d^r y}{dx^r}$ with regard to x in $e^{\frac{m}{x}}$, and the last with regard to x in $x^{n-r-p-2}$, we find

$$-(-1)^p \frac{r(r-1)\dots(r-p+1)}{1.2\dots p} m^{p+1}(n-p-1)\dots(n-r)\,x^{n-r-p-3}e^{\frac{m}{x}}$$

$$+(-1)^{p+1}\frac{r(r-1)\dots(r-p)}{1.2\dots(p+1)} m^{p+1}(n-p-2)\dots(n-r)(n-r-p-2)x^{n-r-p-3}e^{\frac{m}{x}}.$$

On omitting all common factors $\left(\text{including } \frac{1}{p+1}\right)$ we come to

$$(n-p-1)(p+1)+(n-r-p-2)(r-p)=(n-r-1)(r+1);$$

and therefore the general term in $\frac{d^{r+1}y}{dx^{r+1}}$ assumes the form

$$(-1)^{p+1}\frac{(r+1)r\dots(r-p+1)}{1.2\dots(p+1)} m^{p+1}(n-p-2)\dots(n-r-1)\,x^{n-r-p-3}e^{\frac{m}{x}},$$

corresponding to the last of the two general terms in $\frac{d^r y}{dx^r}$.

But for $r=n$, we have

$$\frac{d^n y}{dx^n}=(-1)^n\frac{m^n e^{\frac{m}{x}}}{x^{n+1}},$$

consequently the proposed equation will, by the substitution and after division with $x^{n-1}e^{\frac{m}{x}}$, become

$$(-1)^n m^n - a^n = 0,$$

whence $$m=-a(+1)^{\frac{1}{n}}.$$

Denoting the n roots of $+1$ by $\alpha_1, \alpha_2, \alpha_3 \dots \alpha_n$ we shall have the complete integral

$$y=\Sigma c_r x^{n-1}e^{-\frac{\alpha_r a}{x}},$$

Σ being the sum of terms corresponding to $r=1, 2, 3\dots n$.

Frederiksberg, near Copenhagen, Denmark,
July 1, 1866.

ON SOME SPECIAL FORMS OF CONICS.

By the Rev. GEORGE SALMON.

MY name having been frequently referred to in Mr. Taylor's article, in the last number, with the above title, it may be proper that I should state my views on what seems to me a very simple matter. If it were not that it is objectionable to multiply names without necessity, it would be convenient if in the same manner that we are able to distinguish between a quadrangle and a quadrilateral, so we had different names for "conic" considered as the geometrical interpretation of the general Cartesian, and of the general tangential equation of the second degree. When the discriminant vanishes, the former is always said to denote two right lines: unless we abandon the whole principle of reciprocity, we must say that when the discriminant vanishes, the latter denotes two points. In interpreting a Cartesian equation, we must look to the locus of the points of the system: in interpreting a tangential equation we are to look to the envelope of the lines of the system. In the case of a Cartesian equation of the second degree whose discriminant vanishes, the locus of the points of the system is two right lines; but every line passing through the intersection of these two lines is entitled (and that doubly) to be regarded as a line of the system. We give our name to the system, however, from the *locus*. So in the reciprocal case, all the lines of the system pass through one or other of two points, while every* point on the line joining these points is to be regarded as doubly a point of the system. We must here, however, give our name from the *envelope*, and say that the equation denotes two points.

A system of two points cannot be expressed by a Cartesian equation, nor a system of two lines by a tangential equation. If we determine by the usual rules the Cartesian equation corresponding to a tangential equation denoting two points, we get a perfect square, namely the square of the equation of the line joining the two points, this line, as already explained, containing all the points of the system; but all trace of the two points themselves has disappeared. And the same thing

* *Every* point on the line, and not merely those which lie between the two points. It is incorrect therefore to speak of the locus as a terminated right line. We might receive this impression from considering the line as the limit of ellipses described with the two points as foci; but the line is equally the limit of hyperbolas having the same foci.

occurs if we try to form the tangential equation answering to a Cartesian equation representing two right lines. I refer to my *Conics*, p. 70, for my reasons for not holding with Mr. Taylor that the Cartesian equation of one point is of the second degree; or that of a system of two points, of the fourth degree. It might as well be said that the Cartesian equation of a single point is of the fourth degree, viz. $(x-a)^4+(y-b)^4=0$. But, in truth, in the Cartesian system points are not represented by equations.

Trinity College, Dublin,
Oct. 10, 1866.

Addition by Prof. Cayley.

An equivalent answer to Mr. Taylor's objection is as follows: A conic *quà* curve of the second order and second class cannot degenerate either into a pair of lines or a pair of points; *quà* curve of the second order it can degenerate into a pair of lines but not into a pair of points; *quà* curve of the second class it can degenerate into a pair of points but not into a pair of lines.

ON THE PROBLEM OF THE FIFTEEN SCHOOL GIRLS.

By J. Power, M.A., Fellow of Clare College, Cambridge.

1. This difficult problem may be enunciated as follows:

To arrange fifteen school girls in parties of three for seven consecutive days' walk, so that no two girls may walk together more than once during the seven days.

This problem was first proposed to me by my friend Jonathan Cape, F.R.S., late Professor of Mathematics at Addiscombe. After many trials I gave it up as hopeless; I had indeed no difficulty in obtaining processions for five days, but could not get a step further. I soon however received from Mr. Cape two solutions of his own, founded on a very ingenious principle. Taking eight of the girls, and calling them 1, 2, 3, 4, 5, 6, 7, 8; he formed of these all the *duads* possible, being of course in number $\frac{8\cdot 7}{2}$ or 28. These he arranged in seven octades, each presenting four rows of two in a row. Then came the real difficulty, to place marginally the seven remaining symbols 9, 10, 11, 12, 13, 14, 15, so as to complete the processions in the manner

required. This he accomplished in two ways, and sent me his results with his mode of operating, always adopting the principle of *taking first the least numeral available.*

2. I then sent the enunciation of the problem to my friend Joseph Horner, M.A., Vicar of Everton, and formerly Fellow of Clare College. The problem was quite new to him, but he very soon returned me, not indeed a complete solution, but an ingeniously constructed table of 35 triads of the proper character, sufficing in number for the seven days' walks. These were apparently different from Cape's, but I managed to put them together for the seven processions. I sent this table of 35 triads to Cape, who soon discovered, greatly to his surprise, that if, instead of 1, 2, 3, 4, 5, 6, 7, 8 he used for his 28 duads 8, 9, 10, 11, 12, 13, 14, 15, and if, in place of his former seven 9, 10, 11, 12, 13, 14, 15, he wrote 1, 2, 3, 4, 5, 6, 7; his own triads became identical with Horner's. The reason of this identity lay in the circumstance, that both of them had adopted the same principle of always taking first the least available numeral; without which or something similar, the identity would have been a most singular coincidence.

3. I will first give Horner's 35 triads, and afterwards, with the change of symbols just alluded to, Cape's 28; and then I will extend these 28 to 35 in the manner subsequently adopted by Cape himself, after I had sent him Horner's 35. The identity of the two sets will be at once apparent.

Horner's Table of the 35 *Triads.*

1 2 3						
. 4 5	2 4 6	3 4 7				
. 6 7	. 5 7	. 5 6				
1 8 9	2 8 10	3 8 11	4 8 12	5 8 13	6 8 14	7 8 15
. 10 11	. 9 11	. 9 10	. 9 13	. 9 12	. 9 15	. 9 14
. 12 13	. 12 14	. 12 15	. 10 14	. 10 15	. 10 12	. 10 13
. 14 15	. 13 15	. 13 14	. 11 15	. 11 14	. 11 13	. 11 12

The first column flows immediately from the natural series

1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15,

the leader 1 being understood as preceding each of the seven duads 2 3, 4 5, 6 7, &c. 14 15, so that the series being given the column is known; (by the way, it is useful to observe that conversely, the first column being given, the

series is known). In the second column, 2 4 6 is the first triad next available, since 2 has already walked with 3, and 4 with 5.

The triad 2 5 7 is the next available, because 2 has walked with 6, and 5 with 4.

The same law is observed throughout all the seven columns.

4. The first arrangement of these 35 triads requires no little care and thought, so as to avoid *recurrences*. But, when the table is once correctly formed, it is easy to seize a law of symmetry, by which it may be reproduced at any time as fast as the pen can write.

We may regard the first column as a triad 1, 2, 3, followed by three quaternions 4, 5, 6, 7; 8, 9, 10, 11; 12, 13, 14, 15.

The triad and first quaternion, reading always from left to right, give in their proper order the *leaders* of the seven columns 1, 2, 3, 4, 5, 6, 7.

The second quaternion $\begin{Bmatrix} 8, & 9 \\ 10, & 11 \end{Bmatrix}$ gives in proper order the middle lines 8, 9, 10, 11, read vertically downwards, of the fourth, fifth, sixth and seventh columns of the table.

The third quaternion $\begin{Bmatrix} 12, & 13 \\ 14, & 15 \end{Bmatrix}$, written vertically *downwards*, forms the third vertical line of column fourth, and written vertically *upwards* it forms the third vertical line of the seventh column.

The third vertical lines of the fifth and sixth columns are formed by what may be called the *juxta inversion* of the third lines in the fourth and seventh columns respectively.

As regards the duads of the second and third columns, it may be observed, that their respective quaternions are formed from the corresponding quaternions of the first column, by placing horizontally the direct and cross junctions of the first, thus

$$ab,\ ac,\ ad,$$
$$cd,\ bd,\ bc.$$

Perhaps the whole will be apprehended more clearly, if we write in place of the symbols

1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15,

the following

$$a_1,\ a_2,\ a_3,\ b_1,\ b_2,\ b_3,\ b_4,\ c_1,\ c_2,\ c_3,\ c_4,\ d_1,\ d_2,\ d_3,\ d_4,$$

giving the table as follows:

$a_1a_2a_3$						
b_1b_2	$a_2b_1b_3$	$a_3b_1b_4$				
b_3b_4	b_2b_4	b_2b_3				
c_1c_2	c_1c_3	c_1c_4	$b_1c_1d_1$	$b_2c_1d_2$	$b_3c_1d_3$	$b_4c_1d_4$
c_3c_4	c_2c_4	c_2c_3	c_2d_2	c_2d_1	c_2d_4	c_2d_3
d_1d_2	d_1d_3	d_1d_4	c_3d_3	c_3d_4	c_3d_1	c_3d_2
d_3d_4	d_2d_4	d_2d_3	c_4d_4	c_4d_3	c_4d_2	c_4d_1

5. I will now give Cape's method of deriving the triads, first the 28 and then the remaining 7.

Table of Cape's 28 *Triads.*

8 9..1						
8 10..2	9 10..3					
8 11..3	9 11..2	10 11..1				
8 12..4	9 12..5	10 12..6	11 12..7			
8 13..5	9 13..4	10 13..7	11 13..6	12 13..1		
8 14..6	9 14..7	10 14..4	11 14..5	12 14..2	13 14..3	
8 15..7	9 15..6	10 15..5	11 15..4	12 15..3	13 15..2	14 15..1

Here the duads are first formed from 8, 9...15, and the leaders 1, 2...7 afterwards annexed, the law of taking first the least numeral available, being observed both in the formation of the duads and in the annexation of the leaders.

Cape found that this table required less time to construct it than Horner's did, but my own experience is quite the contrary, and I find that the easiest way of constructing Cape's table is to construct Horner's first, and thence learn what leaders to annex to the duads in Cape's. When I asked Cape how he got the remaining 7 triads, as his table contained only 28, he replied most ingeniously to the following effect. As we divide 1, 2, 3, 4, 5, 6, 7, 8, 9...15, into a less and greater half 1...7 and 8...15, let us pursue the same plan with 1, 2, 3, 4, 5, 6, 7; thus 1, 2, 3; 4, 5, 6, 7; we get

4 5..1		
4 6..2	5 6..3	
4 7..3	5 7..2	6 7..1

In like manner 1, 2, 3 consists of the duad 2, 3, with the leader 1 annexed; the number of Cape's triads is therefore on the whole $28+6+1$ or 35, and they are clearly identical with Horner's, and may be arranged under their respective leaders like his.

6. Cape called my attention to a curious law in the order of the seven leaders as they occur in his table

1
2 3
3 2 1
4 5 6 7
5 4 7 6 1
6 7 4 5 2 3
7 6 5 4 3 2 1.

If from any numeral 7 in the table you go vertically upwards or horizontally to the right, you pass through the same series of figures. The same occurs if you go vertically downwards and horizontally to the left. I would observe, further, that there is a cross relation between the first and second column, by which the second is easily inferred from the first; the process is similar to what I have called elsewhere *juxta-inversion*, thus $\begin{matrix}2\,3\\3\,2\end{matrix},\ \begin{matrix}4\,5\\5\,4\end{matrix},\ \begin{matrix}6\,7\\7\,6\end{matrix}$. The diagonal in which the sevens lie is also remarkable, so is the line of fours, so are the two *triangular segments*, viz.

1   1
2 3   2 3
3 2 1 initial, and 3 2 1 terminal.

Cutting off these, the remaining *square* consists of 4, 5, 6, 7, which numerals occur in reversed order on the opposite sides, and the two diagonals consist of fours and sevens. The whole figure lies symmetrically on each side the diagonal of sevens. These curious relations, if remembered, facilitate materially the formation of Cape's table of triads.

As regards the small supplemental table, the recurrence of

1
2 3
the triangle 3 2 1 cannot fail to be observed. Nor is the law of symmetry extinct even in the monad 1 which belongs to the remaining triad 2 3...1.

On Cape's curious configuration of leaders.

I have had the perseverance to carry out the table, assigning the order of the thirty-two leaders 1, 2...31, to be annexed to the duads arising from the thirty-two symbols 32, 33...63, which duads are in number 496 (see pp. 242, 243). The result is so symmetrical, and shows the laws of configuration so clearly, that I am induced to exhibit it here.

1																														
2	3																													
3	2	1																												
4	5	6	7																											
5	4	7	6	1																										
6	7	4	5	2	3																									
7	6	5	4	3	2	1																								
8	9	10	11	12	13	14	15																							
9	8	11	10	13	12	15	14	1																						
10	11	8	9	14	15	12	13	2	3																					
11	10	9	8	15	14	13	12	3	2	1																				
12	13	14	15	8	9	10	11	4	5	6	7																			
13	12	15	14	9	8	11	10	5	4	7	6	1																		
14	15	12	13	10	11	8	9	6	7	4	5	2	3																	
15	14	13	12	11	10	9	8	7	6	5	4	3	2	1																
16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31															
17	16	19	18	21	20	23	22	25	24	27	26	29	28	31	30	1														
18	19	16	17	22	23	20	21	26	27	24	25	30	31	28	29	2	3													
19	18	17	16	23	22	21	20	27	26	25	24	31	30	29	28	3	2	1												
20	21	22	23	16	17	20	19	28	29	30	31	24	25	26	27	4	5	6	7											
21	20	23	22	17	16	19	18	29	28	31	30	25	24	27	26	5	4	7	6	1										
22	23	20	21	18	19	16	17	30	31	28	29	26	27	24	25	6	7	4	5	2	3									
23	22	21	20	19	18	17	16	31	30	29	28	27	26	25	24	7	6	5	4	3	2	1								
24	25	26	27	28	29	30	31	16	17	18	19	20	21	22	23	8	9	10	11	12	13	14	15							
25	24	27	26	29	28	31	30	17	16	19	18	21	20	23	22	9	8	11	10	13	12	15	14	1						
26	23	24	25	30	31	28	29	18	19	16	17	22	23	20	21	10	11	8	9	14	15	12	13	2	3					
27	26	25	24	31	30	29	28	19	18	17	16	23	22	21	20	11	10	9	8	15	14	13	12	3	2	1				
28	29	30	31	24	25	26	27	20	21	22	23	16	17	18	19	12	13	14	15	8	9	10	11	4	5	6	7			
29	28	31	30	25	24	27	26	21	20	23	22	17	16	19	18	13	12	15	14	9	8	11	10	5	4	7	6	1		
30	31	28	29	26	27	24	25	22	23	20	21	18	19	16	17	14	15	12	13	10	11	8	9	6	7	4	5	2	3	
31	30	29	28	27	26	25	24	23	22	21	20	19	18	17	16	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1

We may remark how every fourth odd number 3, 7, 11, 15 is repeated diagonally upwards. Every fourth even number 4, 8, 12, &c. is repeated diagonally downwards. Every fourth even number of the form 2^n is repeated diagonally, for a number steps represented by its magnitude.

Thus	2	descends	twice,
	4	„	four times,
	8	„	eight times,
	16	„	sixteen times,
	&c.		

The interruptions occur at the boundaries of squares.

The general configuration in duplicate quaternions of the form $\begin{matrix} a & b \\ b & a \end{matrix}$, and the law according to which these quaternions are repeated diagonally with the ascending and descending numbers, may also be remarked.

In every quaternion $b = a + 1$.

Placing the table before us we may *weave* anauthistic* triads as fast as we please in Cape's manner, and to any extent we please. For example,

$A\ B$ 1						
... C 2	$B\ C$ 3					
... D 3	... D 2	$C\ D$ 1				
... E 4	... E 5	... E 6	$D\ E$ 7			
... F 5	... F 4	... F 7	... F 6	$E\ F$ 1		
... G 6	... G 7	... G 4	... G 5	... G 2	$F\ G$ 3	&c.

7. I would observe further on the subject of Cape's tables of triads, that his mode of divisions into a lesser and a greater half, which may be thus represented

$$1'\ 2\ 3'\ 4\ 5\ 6\ 7'\ 8\ 9\ 10\ 11\ 12\ 13\ 14\ 15',$$

gives the following law of the successive *extremes*, 1, 3, 7, 15, 31, 63, 127, 255, the general term of which is $2^n - 1$. These are alternately divisible by 3; in fact if n be even, $2^n - 1$ is of the form $4^m - 1$; and

$$\frac{4^m - 1}{3} = \frac{4^m - 1}{4 - 1} = 4^{m-1} + 4^{m-2} + \ldots + 1.$$

We may hence infer the probability of innumerable other problems, involving processions in triads like the fifteen school girls. For example, $63 = 3 \times 21$.

The first set of duads, formed in Cape's way, will involve 32, 33...63, the number of which is 32, giving therefore

* From α and αὖθις rursus.

$\dfrac{32 \times 31}{2}$ or 31×16, that is 496 duads, which will have to be converted into triads by the annexation of 31 leaders 1, 2...31. The next subordinate set of duads will involve 16, 17...31, the number of which is 16, giving 8×15 or 120 duads, and therefore 120 triads.

To these we must add the successive series already found, $28 + 6 + 1$, making the whole number of triads

$$1 + 6 + 28 + 120 + 496 = 651,$$

which being divided by 21 gives 31 for quotient.

We see then that these 651 triads, if successfully completed by the annexation of leaders to the duads, would exactly suffice for the formation of 31 processions arranged in 21 rows of three in a row, *i.e.* for a month of 31 days instead of a week of 7 days. In the same way we should find, that a school of 255 girls might walk in 85 rows of 3 for 127 days without any two walking together twice.

There is little doubt that the law would admit of generalisation. The series of numbers, representing the days' walks 7, 31, 127, *i.e.* $2^3 - 1$, $2^5 - 1$, $2^7 - 1$, show that the general form is $2^{2n-1} - 1$. Thus the series 1, 3, 7, 15, 31, 63, 127, 255 involves the enunciation of all the problems possible.

A school of	3	may walk	1 day,
"	15	"	7 days,
"	63	"	31 days,
"	255	"	127 days, and so on.

8. Returning once more to Horner's table, I ought to mention the ingenious way in which he first arrived at it. It was by arranging the fifteen symbols in the form of a large quindecagon, and joining the symbols in successive triangles, in such a way that no side of any triangle should be used again for another triangle. As the same process applies to a heptagon of seven symbols, I will give the figure for that case, which will serve as a sufficient illustration of the other (see fig. 18).

PROB. I. To arrange all the processions, &c.

We may regard separately the *leading triads* of Horner's table, seven in number, 123, 145, 167, 246, 257, 347, 356 and the 28 triads

1 8 9	2 8 10	3 8 11	4 8 12	5 8 13	6 8 14	7 8 15
10 11	9 11	9 10	9 13	9 12	9 15	9 14
12 13	12 14	12 15	10 14	10 15	10 12	10 13
14 15	13 15	13 14	11 15	11 14	11 13	11 12

If we compare any two of the former seven, we find that the pair have always one symbol in common, from whence it immediately follows that these seven triads must go on seven separate days.

As the order in which the triads on any day are arranged is immaterial, we may place the above seven leading triads at the head of the seven days' processions in the order above written, viz.:

1st day.	2nd day.	3rd day.	4th day.	5th day.	6th day.	7th day.
1 2 3	1 4 5	1 6 7	2 4 6	2 5 7	3 4 7	3 5 6

The process of completing the processions in every possible way, by help of the above table of twenty-eight triads, is extremely easy. In fact the three figures, of which each of the leading triads consists, at once exclude three of the columns, and leave us only four to choose from. I will take only one heading as an example, say 2 4 6.

The second, fourth and sixth columns are all excluded, and we have to fill up the procession from the first, third, fifth and seventh. It matters not in what order we take the columns; let us take them in the order of their precedence. The first column presents four triads, each of which may be written under 2 4 6, making four varieties to begin with. Let us choose one of them, say 1 8 9. Then looking at the third column, we see that the first two triads 3 8 11, 3 9 10 are excluded by the previous 8 and 9, leaving us the choice of 3 12 15, 3 13 14; and producing, with the former four varieties, a compound variety of 8.

Again, looking at the fifth and seventh columns, we have no choice, but under $\begin{matrix} 1 & 8 & 9 \\ 3 & 12 & 15 \end{matrix}$ to write $\begin{matrix} 5 & 11 & 14 \\ 7 & 10 & 13 \end{matrix}$, and under $\begin{matrix} 1 & 8 & 9 \\ 3 & 13 & 14 \end{matrix}$ to write $\begin{matrix} 5 & 10 & 15 \\ 7 & 11 & 12 \end{matrix}$, the varieties remaining 8 as before; the two processions involving 1 8 9 are, therefore

2 4 6	2 4 6
1 8 9	1 8 9
3 12 15	3 13 14
5 11 14	5 10 15
7 10 13	7 11 12.

The other varieties are found with equal facility. On this principle the following table is completed:

Table shewing the 56 *processions, being all that can be derived from one set of* 35 *triads.*

.	1	2	3	4	5	6	7	8	
	1 2 3	1 2 3	1 2 3	1 2 3	1 2 3	1 2 3	1 2 3	1 2 3	
	4 8 12	4 8 12	4 9 13	4 9 13	4 10 14	4 10 14	4 11 15	4 11 15	
I.	5 10 15	5 11 14	5 10 15	5 11 14	5 8 13	5 9 12	5 8 13	5 9 12	I.
	6 11 13	6 9 15	6 8 14	6 10 12	6 9 15	6 11 13	6 10 12	6 8 14	
	7 9 14	7 10 13	7 11 12	7 8 15	7 11 12	7 8 15	7 9 14	7 10 13	
	1 4 5	1 4 5	1 4 5	1 4 5	1 4 5	1 4 5	1 4 5	1 4 5	
	2 8 10	2 8 10	2 9 11	2 9 11	2 12 14	2 12 14	2 13 15	2 13 15	
II.	3 12 15	3 13 14	3 12 15	3 13 14	3 8 11	3 9 10	3 8 11	3 9 10	II.
	6 11 13	6 9 15	6 8 14	6 10 12	6 9 15	6 11 13	6 10 12	6 8 14	
	7 9 14	7 11 12	7 10 13	7 8 15	7 10 13	7 8 15	7 9 14	7 11 12	
	1 6 7	1 6 7	1 6 7	1 6 7	1 6 7	1 6 7	1 6 7	1 6 7	
	2 8 10	2 8 10	2 9 11	2 9 11	2 12 14	2 12 14	2 13 15	2 13 15	
III.	3 12 15	3 13 14	3 12 15	3 13 14	3 8 11	3 9 10	3 8 11	3 9 10	III.
	4 9 13	4 11 15	4 10 14	4 8 12	4 9 13	4 11 15	4 10 14	4 8 12	
	5 11 14	5 9 12	5 8 13	5 10 15	5 10 15	5 8 13	5 9 12	5 11 14	
	2 4 6	2 4 6	2 4 6	2 4 6	2 4 6	2 4 6	2 4 6	2 4 6	
	1 8 9	1 8 9	1 10 11	1 10 11	1 12 13	1 12 13	1 14 15	1 14 15	
IV.	3 12 15	3 13 14	3 12 15	3 13 14	3 8 11	3 9 10	3 8 11	3 9 10	IV.
	5 11 14	5 10 15	5 8 13	5 9 12	5 10 15	5 11 14	5 9 12	5 8 13	
	7 10 13	7 11 12	7 9 14	7 8 15	7 9 14	7 8 15	7 10 13	7 11 12	
	2 5 7	2 5 7	2 5 7	2 5 7	2 5 7	2 5 7	2 5 7	2 5 7	
	1 8 9	1 8 9	1 10 11	1 10 11	1 12 13	1 12 13	1 14 15	1 14 15	
V.	3 12 15	3 13 14	3 12 15	3 13 14	3 8 11	3 9 10	3 8 11	3 9 10	V.
	4 10 14	4 11 15	4 9 13	4 8 12	4 10 14	4 11 15	4 9 13	4 8 12	
	6 11 13	6 10 12	6 8 14	6 9 15	6 9 15	6 8 14	6 10 12	6 11 13	
	3 4 7	3 4 7	3 4 7	3 4 7	3 4 7	3 4 7	3 4 7	3 4 7	
	1 8 9	1 8 9	1 10 11	1 10 11	1 12 13	1 12 13	1 14 15	1 14 15	
VI.	2 12 14	2 13 15	2 12 14	2 13 15	2 8 10	2 9 11	2 8 10	2 9 11	VI.
	5 10 15	5 11 14	5 8 13	5 9 12	5 11 14	5 10 15	5 9 12	5 8 13	
	6 11 13	6 10 12	6 9 15	6 8 14	6 9 15	6 8 14	6 11 13	6 10 12	
	3 5 6	3 5 6	3 5 6	3 5 6	3 5 6	3 5 6	3 5 6	3 5 6	
	1 8 9	1 8 9	1 10 11	1 10 11	1 12 13	1 12 13	1 14 15	1 14 15	
VII.	2 12 14	2 13 15	2 12 14	2 13 15	2 8 10	2 9 11	2 8 10	2 9 11	VII.
	4 11 15	4 10 14	4 9 13	4 8 12	4 11 15	4 10 14	4 9 13	4 8 12	
	7 10 13	7 11 12	7 8 15	7 9 14	7 9 14	7 8 15	7 11 12	7 10 13	
	1	2	3	4	5	6	7	8	

9. The table is arranged so that any one of the fifty-six processions may be singled out, so to speak, by its latitude and longitude, thus I_1, II_2, &c.

Abstracting from the leading triads in these processions, we may remark that any two processions we choose to take, have necessarily two out of four leaders in common. Thus I_1 and II_1 have the leaders 6 and 7 in common, and in this instance they are followed by like duads 11 13, 9 14, so that I_1 and II_1 cannot enter into any solution together. Neither can I_1 and II_7 on account of the one common triad 7 9 14. But I_1 and II_2 have no triad in common, and may therefore enter together to form a solution. In testing this compati-

bility, we need not trouble ourselves about examining any triads, but those which present a common leader. This readily attracts the eye and facilitates the operation.

10. PROB. II. To determine all the solutions which can be derived from one set of triads, and to assign their number.

For the first day we have the choice of the eight processions $I_1, I_2 \ldots I_8$. We write down therefore a *variety-factor* 8 and choose any one of them, say I_1. Write this fully out on a moveable slip of paper and compare it very carefully with

$$II_1, II_2, II_3, II_4, II_5, II_6, II_7, II_8,$$

and strike the pen through those which present any common triad or triads. It will be found that II_1, II_6, and II_7 are thus cancelled, but there is a free choice between the other five,

$$II_2, II_3, II_4, II_5, II_8.$$

Write down, therefore, a second variety-factor 5 and choose one of them, say II_2. It is essential to observe that, had our first choice been any other of the eight for the first day, in any case three would be cancelled and five would remain available for the second day, so that we have the same variety-factor 5 in every case.

Add the procession II_2, written down in full, to the moveable slip, and compare the two I_1, II_2 with each of the following

$$III_1, III_2, III_3, III_4, III_5, III_6, III_7, III_8,$$

and striking out all that have triads in common with I_1 or II_2, we find, as above, all cancelled but three

$$III_3, III_6, III_7.$$

Write down therefore a new variety-factor 3, and choose one, as III_3, adding it to the moveable slip.

It is easy to convince ourselves even without verification by actual trial, that a like freedom of choice would remain had we chosen differently for the first and second days; in fact, the symmetry of the table of processions, in which all the triads are similarly involved, leaves no doubt on the subject to a mind of ordinary apprehension. A similar remark will apply at each succeeding step, and it will be sufficient to have made it here once for all.

Carrying the moveable slip, having I_1, II_2, III_8 written out at full, over

$$IV_1,\ IV_2,\ IV_3,\ IV_5,\ IV_6,\ IV_7,\ IV_8,$$

and striking out as above, we find all cancelled but IV_6 and IV_7, for which we record a variety-factor 2; and choosing IV_6, we find in like manner on the fifth, sixth, and seventh days as follows:

$$V_1,\quad V_2,\quad V_3,\quad V_4,\quad V_5,\quad V_6,\quad V_7,\quad V_8,$$
$$VI_1,\quad VI_2,\quad VI_3,\quad VI_4,\quad VI_5,\quad VI_6,\quad VI_7,\quad VI_8,$$
$$VII_1,\quad VII_2,\quad VII_3,\quad VII_4,\quad VII_5,\quad VII_6,\quad VII_7,\quad VII_8,$$

each giving no alternative of choice but V_7, VI_4, VII_1 on the respective days; and making the variety-factor 1 in each case.

The solution at which we have arrived, by choosing always the first available triad, is

I_1	II_2	III_8	IV_6	V_7	VI_4	VII_1
1 2 3	1 4 5	1 6 7	2 4 6	2 5 7	3 4 7	3 5 6
4 8 12	2 8 10	2 9 11	1 12 13	1 14 15	1 10 11	1 8 9
5 10 15	3 13 14	3 12 15	3 9 10	3 8 11	2 13 15	2 12 14
6 11 13	6 9 15	4 10 14	5 11 14	4 9 13	5 9 12	4 11 15
7 9 14	7 11 12	5 8 13	7 8 15	6 10 12	6 8 14	7 10 13

but the whole number of solutions, of which this is only one, is

$$8\times5\times3\times2\times1\times1\times1=240.\quad \text{Q.E.I.}$$

Appendix to the Demonstration of the 240 *Solutions belonging to one set of triads.*

As the derivation of these solutions requires time and very careful work, I may as well record here some which I have worked out.

I	II	III	IV	V	VI	VII
1	2	3	6	7	4	1
2	1	5	4	6	8	2
3	1	6	4	5	2	8
4	1	4	7	6	3	2
5	1	4	6	7	4	1
6	2	5	1	6	8	4
7	2	5	1	8	4	6
8	1	4	6	7	3	2

Remarks.

The Arabic numerals are to be regarded as suffixes to the Roman numerals representing the days.

These 8 solutions give for every one of the eight $I_1, I_2, I_3, I_4, I_5, I_6, I_7, I_8,$ a solution on the principle of taking first the earliest available, in other words of keeping always nearest to the left-hand margin of the table.

I	II	III	IV	V	VI	VII
1	2	3	6	7	4	1
1	3	2	6	6	3	2
1	4	1	7	6	3	2
1	5	1	4	6	8	2
1	8	1	4	5	8	1
1	2	3	6	7	4	1
.	.	6	1	7	4	6
.	.	7	1	6	8	3
1	2	3	6	7	4	1
.	.	.	7	6	2	3

2	1	5	4	6	8	2
2	4	5	3	6	7	2
2	6	2	3	7	6	2
2	7	2	8	1	6	3
2	8	2	3	7	1	6
2	1	5	4	6	8	2
.	.	6	2	7	4	6
.	.	7	2	6	8	3
2	1	5	4	6	8	2
.	.	.	2	7	4	6

A repetition.

These 5 solutions exhibit the 5 variations on the second day, all preceded by I_1, and otherwise obeying the law of earliest first.

Repetition.

These 3 solutions exhibit the 3 varieties on the third day all worked out successfully as before.

Repetition.

These 2 exhibit the 2 varities on the fourth day worked out successfully.

Repetition.

These tables represent another set of solutions all preceded by 1_2, and like the former exhibit the varieties.

Repetition.

Repetition.

Here are exhibited in all 22 different solutions.

11. PROB. III. To determine how many different sets of triads may be formed of fifteen symbols.

If under the series

1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15,

we write any permutation of the order of these same symbols, as, for example

3, 2, 1, 6, 5, 4, 9, 8, 7, 12, 11, 10, 15, 14, 13,

and if in place of each of the upper symbols we substitute the corresponding symbol of the lower series in Horner's table of triads, we shall obtain, generally speaking, a set of different triads, and this may be done in 1.2.3.4...15 different ways, apparently leading to as many different sets of triads.

It might, however, happen that some of these different substitutions gave the same identical triads over and over again. This is what will really occur, and I am going to show that to any one set of triads there will correspond exactly 1.2.3.4.5.6.7 permutations of the order 1, 2, 3, ...15 all producing one and the same set of triads.

Take from Horner's table of triads, or from any solution involving those triads, any one of the thirty-five triads, say 5, 9, 12.

As this is one of 35, we record a variety-factor 35.

Again the triad 5 9 12 may be written in six different ways: 5 9 12; 9 5 12, &c. Record another factor 6, and choose one of the triads as 9 5 12, it matters not which; since 9 must enter into all the seven processions, there are six other triads into which 9 enters; take any one of them as 9 6 15 recording the factor 6, and again a factor 2, inasmuch as it is a matter of free choice whether we write 9 6 15 or 9 15 6; let us choose 9 6 15. We may then begin a new Horner's table, thus $\begin{array}{ccc} 9 & 5 & 12 \\ & 6 & 15 \end{array}$ without change of triads as far as we have gone.

In the next place look in the old table for the completion of the triads 5 6 ., 5 15 .; we readily find 5 6 3, 5 15 10, and writing the duad 3 10 in our new first column, and proceeding as usual, we get by direct and cross junctions (see No. 4)

9 5 12		
9 6 15	5 6 3	12 3 15
9 3 10	5 15 10	12 6 10

.

these seven triads all agreeing with the old ones.

There are remaining four others of the seven triads into which 9 enters; of these choose any one at random, say 9 1 8, recording a variety-factor 4; and since it may be written 9 1 8 or 9 8 1, record another factor 2. Let us choose 9 1 8, and look for the completion of 5 1 ., 5 8 .; we find 5 1 4, 5 8 13, and we proceed with our new table thus:

9 5 12						
6 15	5 6 3	12 3 15				
3 10	15 10	6 10				
9 1 8	5 1 4	12 1 13	6 1 .	15 1 .	3 1 .	10 1 .
4 13	8 13	4 8	8 .	8 .	8 .	8 .
. .	. .	. .	4 .	4 .	4 .	4 .
. .	. .	. .	13 .	13 .	13 .	13 .

Complete now the four triads

6 1 .
6 8 .
6 4 .
6 13 .

from the old table, and we readily find

6 1 7
6 8 14
6 4 2
6 13 11

which gives 7, 14, 2, 11, for the last quaternion of the first column, and we now have the whole of a new table

9 5 12							
6 15	5 6 3	12 6 10					
3 10	15 10	3 15					
9 1 8	5 1 4	12 1 13	6 1 7	15 1 14	3 1 2	10 1 11	10 1 11
4 13	8 13	4 8	8 14	8 7	8 11	8 2	
7 14	7 2	7 11	4 2	4 11	4 7	4 14	
2 11	14 11	2 14	13 11	13 2	13 14	13 7	

in which the triads are manifestly identical with the original, and which corresponds to the permutation

9, 5, 12, 6, 15, 3, 10, 1, 8, 4, 13, 7, 14, 2, 11.

To find in how many ways this might be done, we have only to multiply our variety-factors 35.6.6.2.4.2, their product is the same thing as $1.2.3.4.5.6.7 \times 4$. This will be exactly the number of the rows

1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15,

9, 5, 12, 6, 15, 3, 10, 1, 8, 4, 13, 7, 14, 2, 11,

&c. &c.

all of which produce the same identical set of triads.

Exactly the same number of permutation rows will belong to every set of triads. Consequently if t be the number of *different* sets of triads, since to each set correspond $1.2\ldots7 \times 4$ rows of permutations, all producing the same triads, the product $1.2.3.4.5.6.7 \times 4t$ must be the whole number of permutations $1.2.3\ldots15$. Consequently

$$t = \frac{8.9.10.11.12.13.14.15}{4}. \quad \text{Q.E.I.}$$

The same process will apply to arrange the triads of any

given solution in Horner's manner. For example, Kirkman's Solution, *Ladies' Diary*, 1854, is

$a_1a_2a_3$	$a_1b_1c_1$	$a_1d_1e_1$	$a_1b_2d_2$	$a_1c_2e_2$	$a_1b_3e_3$	$a_1c_3d_3$
$b_1b_2b_3$	$a_2b_2c_2$	$a_2d_2e_2$	$a_2b_3d_3$	$a_2c_3e_3$	$a_2b_1e_1$	$a_2c_1d_1$
$c_1c_2c_3$	$a_3d_3e_3$	$a_3b_3c_3$	$a_3c_1e_1$	$a_3b_1d_1$	$a_3c_2d_2$	$a_3b_2e_2$
$d_1d_2d_3$	$b_3d_1e_2$	$d_3b_1c_1$	$b_1c_3e_2$	$c_1b_3d_2$	$b_2c_3d_1$	$c_2b_3e_1$
$e_1e_2e_3$	$c_3d_2e_1$	$e_3b_2c_1$	$d_1c_2e_3$	$e_1b_2d_3$	$e_2c_1d_3$	$d_2b_1e_3$

I find as follows:

H 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15,

K a_1, a_2, a_3, b_1, c_1, e_1, d_1, b_2, d_2, c_2, e_2, b_3, e_3, d_3, c_3,

and substituting the upper for the lower, the solution becomes

1 2 3	1 4 5	1 7 6	1 8 9	1 10 11	1 12 13	1 15 14
4 8 12	2 8 10	2 9 11	2 12 14	2 15 13	2 4 6	2 5 7
5 10 15	3 14 13	3 12 15	3 5 6	3 4 7	3 10 9	3 8 11
7 9 14	12 7 11	14 4 10	4 15 11	5 12 9	8 15 7	10 12 6
6 11 13	15 9 6	13 8 5	7 10 13	6 8 14	11 5 14	9 4 13

These processions are severally identical with

$$\mathrm{I}_1,\ \mathrm{II}_2,\ \mathrm{III}_3,\ \mathrm{VII}_1,\ \mathrm{VI}_4,\ \mathrm{IV}_6,\ \mathrm{V}_7,$$

and the above differs not from my solution

$$\mathrm{I}_1,\ \mathrm{II}_2,\ \mathrm{III}_3,\ \mathrm{IV}_6,\ \mathrm{V}_7,\ \mathrm{VI}_4,\ \mathrm{VII}_1.$$

The series K, viz. a_1, a_2, a_3, &c. is only one of 1.2.3.4.5.6.7.4 equally possible, but the corresponding solutions are not all that are possible with the same fifteen symbols. They all belong to only *one* set of triads, and since their number may be shown to be 240×84, the same 240 solutions must occur 84 times over.

12. Prob. IV. To find the whole number of different solutions of which the School-girls problem is capable.

Since each set of triads gives 240 solutions, we have only to multiply the preceding value of t by 240, the result is

$$8.9.10.11.12.13.14.15 \times 60,$$

$$= 15{,}567{,}552{,}000,$$

i.e. 15 billions, 567 millions, and 552 thousands, or something more than 15½ billions, and no two of these solutions can coincide throughout.

Litlington, near Royston,
June 26, 1865.

SOLUTION OF A PARTIAL DIFFERENTIAL EQUATION.

By Prof. Schläfli.

The equation considered is

$$a\left(y\frac{dw}{dz}-z\frac{dw}{dy}\right)^2+b\left(z\frac{dw}{dx}-x\frac{dw}{dz}\right)^2+c\left(x\frac{dw}{dy}-y\frac{dw}{dx}\right)^2=1.$$

Writing for shortness

$$\left(\frac{dw}{dx},\ \frac{dw}{dy},\ \frac{dw}{dz}\right)=(p,\ q,\ r),$$

and denoting moreover by (P, Q, R) the determinants

$$\begin{vmatrix} x, & y, & z \\ p, & q, & r \end{vmatrix},$$

the proposed equation becomes

$$aP^2+bQ^2+cR^2=1,$$

or, what is the same thing,

$$U=\begin{vmatrix} aP, & bQ, & cR \\ x, & y, & z \\ p, & q, & r \end{vmatrix}=1.$$

If we distinguish by brackets the minors of this matrix from the corresponding elements, we obtain

$$\tfrac{1}{2}dU=(x)\,dx+(y)\,dy+(z)\,dz+(p)\,dp+(q)\,dq+(r)\,dr=0.$$

Forming the total differential equations on the integration of which depends the solution of the problem according to Pfaff's rule, we see that dx, dy, dz, dp, dq, dr are respectively proportional to (p), (q), (r), $-(x)$, $-(y)$, $-(z)$, and consequently $dw\ (=pdx+qdy+rdz)$ to $U=1$, that is to say, $dx=(p)\,dw$, &c. Therefore the matrix

$$\begin{matrix} Pdw, & Qdw, & Rdw \\ -dp, & -dq, & -dr \\ dx, & dy, & dz \end{matrix}$$

shows the minors of U multiplied by dw. Let us combine, first, lines of both matrices, and, secondly, columns of both matrices, and take the sums of binary products. According as the lines or columns combined are corresponding ones or not, we shall obtain dw or 0.

We obtain, first, $-(xdp + ydq + zdr) = dw$, and subtracting this from $pdx + qdy + rdz = dw$, we get

$$d(xp + yq + zr) = 0;$$

again $\quad xdx + ydy + zdz = 0, \;\; pdp + qdq + rdr = 0.$

Therefore Σxp, Σx^2, Σp^2 are constants; consequently also $\Sigma x^2 . \Sigma p^2 - (\Sigma xp)^2 = \Sigma P^2$. Hence we may write

$$x^2 + y^2 + z^2 = m^2 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

$$P^2 + Q^2 + R^2 = N^2 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2),$$

$$xp + yq + zr = N \cot \zeta \;\ldots\ldots\ldots\ldots\ldots (3),$$

(where m, N, ζ denote arbitrary constants of integration). A mere consequence hereof is

$$p^2 + q^2 + r^2 = \left(\frac{N}{m \sin \zeta}\right)^2.$$

Secondly, we obtain $aP^2dw - xdp + pdx = dw$, that is to say, $pdx - xdp = (1 - aP^2)\, dw$, &c.; again

$$bQRdw - ydr + qdz = 0, \quad cQRdw - zdq + rdy = 0,$$

and, by subtraction,

$$(b - c)\, QRdw = dP, \text{ \&c.}$$

On account of $P^2 + R^2 = N^2 - Q^2$, $aP^2 + cR^2 = 1 - bQ^2$ we may consider P, R as given functions of Q, and we shall next confine our attention to the two equations

$$qdy - ydq = (1 - bQ^2)\, dw, \quad (c - a)\, PRdw = dQ.$$

The latter at once affords a fourth integral

$$w = C + \int_{Q=0} \frac{dQ}{(c - a)\, PR} \;\ldots\ldots\ldots\ldots\ldots (4).$$

As regards the former, we need but a slight transformation in order to integrate it. Since

$$\begin{vmatrix} y, & x^2 + y^2 + z^2 \\ q, & xp + yq + zr \end{vmatrix} = zP - xR = N \cot \zeta . y - m^2 q,$$

it becomes

$$yd(zP-xR)-(zP-xR)\,dy=m^2(1-bQ^2)\,dw;$$

and because the identical equation $xP+yQ+zR=0$ leads to $(zP-xR)^2+(Ny)^2=(x^2+z^2)(P^2+R^2)+y^2(N^2-Q^2)=m^2(N^2-Q^2)$, we are at liberty to assume

$$N^2-Q^2=\rho^2,\quad zP-xR=m\rho\cos\psi,\quad y=-\frac{m\rho}{N}\sin\psi.$$

The differential equation will then change into

$$d\psi=N\frac{1-bQ^2}{N^2-Q^2}\,dw,$$

and furnishes us with a fifth integral

$$\psi=n+\int_{Q=0}\frac{N}{(c-a)\,PR}\,\frac{1-bQ^2}{N^2-Q^2}\,dQ\,\ldots\ldots\ldots(5).$$

Now, not only y is known in function of Q, but also x, z are so from the two equations

$$Px+Rz=\frac{m\rho}{N}\,Q\sin\psi,\quad -Rx+Pz=m\rho\cos\psi.$$

We are now to transform the expression

$$pdx+qdy+rdz-dw$$

into one containing the differentials of all the six elements Q, m, n, N, C, ζ. But we will first observe that P, R, $w-C$, $\psi-n$ involve only Q, N, and that the expressions of $\frac{x}{m}$, $\frac{y}{m}$, $\frac{z}{m}$ do not contain m, but only Q, N, n. Hence it is evident that $d\zeta$ cannot occur, that only the term $-dw$ furnishes $-dC$, and that the coefficient of dm becomes

$$\frac{1}{m}(px+qy+rz)=\frac{N}{m}\cot\zeta;$$

moreover it is plain that dQ cannot occur at all, since the integral equations (1), (2), ... (5) satisfy the above system of six differential equations under the supposition of m, n, N, C, ζ being constants. So we are only concerned with n, N. Let δ denote a differentiation only in regard to n, N. Then, from $P\delta P+R\delta R=NdN$, $aP\delta P+cR\delta R=0$, follows

$$z\delta P-x\delta R=N\frac{axP+czR}{(c-a)\,PR}\,dN.$$

Again, because $x\delta x + y\delta y + z dz = 0$, we have

$$\begin{vmatrix} y, & x\delta x + y\delta y + z\delta z \\ q, & p\delta x + q\delta y + r\delta z \end{vmatrix} = y\,(p\delta x + q\delta y + r\delta z)$$

$$= P\delta z - R\delta x$$

$$= \delta\,(zP - xR) - N\frac{axP + czR}{(c-a)\,PR}\,dN.$$

But $\delta\,(zP - xR) = \delta\,(m\rho\cos\psi) = N\dfrac{m\cos\psi}{\rho}\,dN - m\rho\sin\psi\,\delta\psi$

$$= N\frac{zP - xR}{N^2 - Q^2}\,dN + Ny\delta\psi,$$

and

$$N\left(\frac{zP - xR}{N^2 - Q^2} - \frac{axP + czR}{(c-a)\,PR}\right)dN$$

$$= \frac{N(1 - bQ^2)}{(c-a)\,PR\,(N^2 - Q^2)}\,(-xP - zR)\,dN = \frac{d\psi}{dQ}\,yQdN.$$

Therefore

$$p\delta x + q\delta y + r\delta z = Ndn + \left(Q\frac{d\psi}{dQ} + N\frac{d\psi}{dN}\right)dN.$$

We can now write

$$pdx + qdy + rdz - dw = -dC + N\left(\cot\zeta\,\frac{dm}{m} + dn\right) + \Omega dN,$$

where
$$\Omega = Q\frac{d\psi}{dQ} + N\frac{d\psi}{dN} - \frac{d\omega}{dN}.$$

This expression vanishes when $Q = 0$, because then $\psi = n$, $w = C$; so we only need to investigate the value of

$$\frac{d\Omega}{dQ} = Q\frac{d^2\psi}{dQ^2} + \frac{d}{dN}\left(N\frac{d\psi}{dQ} - \frac{d\omega}{dQ}\right).$$

To do this with more ease we put $N^2 - Q^2 = u$, $1 - bQ^2 = uv$, $(c-v)\,(v-a) = V^2$, so that

$$(c-a)\,PR = uV,\quad \frac{d\psi}{dQ} = \frac{Nv}{uV},\quad \frac{d\omega}{dQ} = \frac{1}{uV},$$

and
$$N\frac{d\psi}{dQ} - \frac{d\omega}{dQ} = Q^2\frac{v-b}{uV}.$$

Then

$$\frac{u}{2NQ^2}\,\frac{d\Omega}{dQ} = \frac{u}{2Q}\,\frac{d}{dQ}\left(\frac{v}{uV}\right) + \frac{u}{2N}\,\frac{d}{dN}\left(\frac{v-b}{uV}\right)$$

$$= \left(-u\frac{d}{du} + (v-b)\frac{d}{dv}\right)\left(\frac{v}{uV}\right) + \left(u\frac{d}{du} - v\frac{d}{dv}\right)\left(\frac{v-b}{uV}\right)$$

$$= -\frac{Cu}{V}\,\frac{d}{du}\left(\frac{1}{u}\right) - \frac{C}{uV} = 0.$$

Hence $\Omega = 0$ in general.

The condition $dw = pdx + qdy + rdz$, to be still fulfilled when all six elements Q, m, n, N, C, ζ are considered as variables, has now changed into

$$dC = N\left(\cot\zeta\,.\,\frac{dm}{m} + dn\right) \,\ldots\ldots\ldots\ldots\ldots (6).$$

I think it needless to go further, since the fact that there are only three differentials in (6), shows clearly enough that, in the main, the problem is solved, and, I may add, without applying initial values.

INVESTIGATION OF AN ALGEBRAICAL FORMULA.

By J. C. W. Ellis, M.A., Fellow of Sidney Sussex College.

To prove the formula:

$$\lceil\overset{p}{\overline{1}} = \lceil\overset{p-n}{\overline{n+1}} - \overset{0}{\overline{n}}\rceil\;\lceil\overset{p-n}{\overline{n}} + \overset{1}{\overline{n-1}}\rceil\;\lceil\overset{p-n}{\overline{n-1}} - \overset{2}{\overline{n-2}}\rceil\;\lceil\overset{p-n}{\overline{n-2}} + \&c. \pm \lceil\overset{p-n}{\overline{1}}$$

$+$ if n be even

$-$ if n be odd.

Where $\lceil\overset{p}{\overline{n}}$ represents $\frac{\lfloor n}{\lfloor 1}1^p + \frac{\lfloor n+1}{\lfloor 2}2^p + \frac{\lfloor n+2}{\lfloor 3}3^p + \ldots$ to m terms,

$\overset{0}{\overline{n}}\rceil$ .... $n + (n-1) + (n-2) + \ldots. + 1$,

$\overset{1}{\overline{n}}\rceil$ .... $n\,\overset{1}{\overline{n}}\rceil + (n-1)\,\overset{1}{\overline{n-1}}\rceil + \&c. + \overset{1}{\overline{1}}\rceil$,

....

$\overset{r}{\overline{n}}\rceil$ .... $n\,\overset{r}{\overline{n}}\rceil + (n-1)\,\overset{r}{\overline{n-1}}\rceil + \&c. + \overset{r}{\overline{1}}\rceil$.

Proof:

$$\overset{p}{\lceil\overline{n}} = \frac{\lfloor\underline{n}}{\lfloor\underline{1}}\,1^p + \frac{\lfloor\underline{n+1}}{\lfloor\underline{2}}\,2^p + \frac{\lfloor\underline{n+2}}{\lfloor\underline{3}}\,3^p + \dots$$

$$= \frac{\lfloor\underline{n}}{\lfloor\underline{1}}\,\{(n+1)-n\}\,1^{p-1} + \frac{\lfloor\underline{n+1}}{\lfloor\underline{2}}\,\{(n+2)-n\}\,2^{p-1} + \frac{\lfloor\underline{n+2}}{\lfloor\underline{3}}\,\{(n+3)-n\}\,3^{p-1} + \&c.$$

$$= \frac{\lfloor\underline{n+1}}{\lfloor\underline{1}}\,1^{p-1} + \frac{\lfloor\underline{n+2}}{\lfloor\underline{2}}\,2^{p-1} + \frac{\lfloor\underline{n+3}}{\lfloor\underline{3}}\,3^{p-1} + \&c. - n\,\overset{p-1}{\lceil\overline{n}}$$

$$= \overset{p-1}{\lceil\overline{n+1}} - n\,\overset{p-1}{\lceil\overline{n}};$$

therefore since $\overset{p}{\lceil\overline{n}} = \overset{p-1}{\lceil\overline{n+1}} - n\,\overset{p-1}{\lceil\overline{n}}$,

$$\overset{p}{\lceil\overline{1}} = \overset{p-1}{\lceil\overline{2}} - \overset{p-1}{\lceil\overline{1}} \dots\dots\dots \text{i}$$

$$= \overset{p-2}{\lceil\overline{3}} - (2+1)\,\overset{p-2}{\lceil\overline{2}} + \overset{p-2}{\lceil\overline{1}}$$

$$= \overset{p-2}{\lceil\overline{3}} - \overset{0}{\overline{2}\rceil}\,\overset{p-2}{\lceil\overline{2}} + \overset{1}{\overline{1}\rceil}\,\overset{p-2}{\lceil\overline{1}} \dots\dots\dots \text{ii}$$

$$= \overset{p-3}{\lceil\overline{4}} - (3 + \overset{0}{\overline{2}\rceil})\,\overset{p-3}{\lceil\overline{3}} + (2\,\overset{0}{\overline{2}\rceil} + \overset{1}{\overline{1}\rceil})\,\overset{p-3}{\lceil\overline{2}} - \overset{p-3}{\lceil\overline{1}}$$

$$= \overset{p-3}{\lceil\overline{4}} - \overset{0}{\overline{3}\rceil}\,\overset{p-3}{\lceil\overline{3}} + \overset{1}{\overline{2}\rceil}\,\overset{p-3}{\lceil\overline{2}} - \overset{p-3}{\lceil\overline{1}} \dots\dots\dots \text{iii}$$

$$= \&c.$$

$$= \overset{p-n}{\lceil\overline{n+1}} - \overset{0}{\overline{n}\rceil}\,\overset{p-n}{\lceil\overline{n}} + \overset{1}{\overline{n-1}\rceil}\,\overset{p-n}{\lceil\overline{n-1}} - \dots \pm \overset{p-n}{\lceil\overline{1}},$$

as n is even or odd. Q.E.D.

Hence $1^p + 2^p + 3^p + \dots m^p = \overset{p}{\lceil\overline{1}}$

$$= \overset{1}{\lceil\overline{p}} - \overset{0}{\overline{p-1}\rceil}\,\overset{1}{\lceil\overline{p-1}} + \overset{1}{\overline{p-2}\rceil}\,\overset{1}{\lceil\overline{p-2}} - \overset{2}{\overline{p-3}\rceil}\,\overset{1}{\lceil\overline{p-3}} + \&c.$$

$$= \frac{\lfloor\underline{m+p}}{(p+1)\lfloor\underline{m-1}} - \overset{0}{\overline{p-1}\rceil}\,\frac{\lfloor\underline{m+p-1}}{p\lfloor\underline{m-1}} + \overset{1}{\overline{p-2}\rceil}\,\frac{\lfloor\underline{m+p-2}}{(p-1)\lfloor\underline{m-1}} - \&c.$$

$$= \frac{\lfloor\underline{m+p}}{\lfloor\underline{m-1}}\left[\frac{1}{p+1} - \frac{\overset{0}{\overline{p-1}\rceil}}{p(m+p)} + \frac{\overset{1}{\overline{p-2}\rceil}}{(p-1)(m+p)(m+p-1)} - \&c.\right] \dots\dots\dots \text{iv.}$$

For $\overset{1}{\lceil\overline{p}} = 1.2.3\dots p + 2.3.4\dots(p+1) + \&c.$ to m terms

$$= \frac{m(m+1)\dots(m+p)}{p+1} = \frac{\lfloor\underline{m+p}}{(p+1)\lfloor\underline{m-1}}.$$

From iv. we see that $\dfrac{\overset{p}{\lceil 1}}{m^{p+1}}$ is $=\dfrac{1}{p+1}$ when m is infinite.

Ex. $1^3+2^3+3^3+\ldots+m^3=\overset{3}{\lceil 1}$

$$=\frac{\lfloor m+3}{\lfloor m-1}\left[\frac{1}{4}-\frac{3}{3(m+3)}+\frac{1}{2(m+3)(m+2)}\right]$$

$$=\frac{m(m+1)(m+2)}{4}\left[(m+3)-2\,\frac{2m+3}{m+2}\right]$$

$$=\frac{\{m(m+1)\}^2}{4}=(\overset{1}{\lceil 1})^2.$$

Observing that

$$\overset{p}{n}\rceil=n\,\overset{p-1}{n}\rceil+\overset{p}{\overline{n-1}}\rceil.$$

The calculation becomes very simple, and the table to be used will be

$$\overset{0}{1}\rceil=1,$$
$$\overset{0}{2}\rceil=3,\quad \overset{1}{1}\rceil=1,$$
$$\overset{0}{3}\rceil=6,\quad \overset{1}{2}\rceil=7,\quad \overset{2}{1}\rceil=1,$$
$$\overset{0}{4}\rceil=10,\quad \overset{1}{3}\rceil=25,\quad \overset{2}{2}\rceil=15,\quad \overset{3}{1}\rceil=1,$$
$$\overset{0}{5}\rceil=15,\quad \overset{1}{4}\rceil=65,\quad \overset{2}{3}\rceil=90,\quad \overset{3}{2}\rceil=31,\quad \overset{4}{1}\rceil=1,$$
$$\overset{0}{6}\rceil=21,\quad \overset{1}{5}\rceil=140,\quad \overset{2}{4}\rceil=350,\quad \overset{3}{3}\rceil=301,\quad \overset{4}{2}\rceil=63,\quad \overset{5}{1}\rceil=1,$$
$$\overset{0}{7}\rceil=28,\quad \overset{1}{6}\rceil=266,\quad \overset{2}{5}\rceil=1050,\quad \overset{3}{4}\rceil=1701,\quad \overset{4}{3}\rceil=966,\quad \overset{5}{2}\rceil=127,\quad \overset{6}{1}\rceil=1$$

Thus, by formula iv., $1^6+2^6+3^6+\ldots+m^6$

$$=\frac{\lfloor m+6}{\lfloor m-1}\left[\frac{1}{7}-\frac{15}{6(m+6)}+\frac{65}{5(m+6)(m+5)}\right.$$
$$-\frac{90}{4(m+6)(m+5)(m+4)}+\frac{31}{3(m+6)(m+5)(m+4)(m+3)}$$
$$\left.-\frac{1}{2(m+6)\ldots m+2}\right].$$

THEOREMS RELATING TO THE GROUP OF CONICS PASSING THROUGH FOUR GIVEN POINTS.

By N. M. Ferrers.

Let a group of conics be described, passing through four given points. We propose, first, to prove that a system of straight lines can be found, such that if through the two points in which each conic of the group is cut by any one straight line of the system, an opposite, equal, similar, and similarly situated conic be drawn, the group of conics thus obtained shall intersect in four points; secondly, to investigate the envelope of the system of straight lines; and thirdly, to prove that the locus of each one of the four points thus obtained, is a conic similar and similarly situated to the locus of centres of the group of conics passing through the four given points.

In the first place, suppose the four points to form a convex quadrangle, that is, that each lies without the triangle formed by the other three. Then through these four points two real parabolas can be drawn. Take as co-ordinate axes that diameter of each the tangent at the extremity of which is parallel to the axis of the other. Let the parabolas be represented by the equations

$$S \equiv x^2 + 2fy \ \ + c = 0 \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

$$S' \equiv y^2 + 2g'x + c' = 0 \ldots\ldots\ldots\ldots\ldots\ldots (2),$$

then the group of conics will be represented by the equation

$$S + kS' = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3),$$

the particular conic being identified by the value of the parameter k.

Now, let the equation of any straight line of the system be

$$\xi x + \eta y - 1 = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (4),$$

then, the equation of all conics, similar and similarly situated to (3) and passing through its points of intersection with (4), is

$$S + kS' + 2\lambda(\xi x + \eta y - 1) = 0 \ldots\ldots\ldots\ldots (5).$$

This will be made equal to (3) by assigning a suitable value to λ, which will thus be a function of ξ, η, k, and the parameters in (1) and (2).

And if λ_1, λ_2 be the two specific values of λ corresponding to (1) and (2), the condition of intersection will be

$$\lambda = \lambda_1 + k\lambda_2 \dots\dots\dots\dots\dots\dots\dots\dots (6).$$

We determine λ by the condition that the point midway between the centres of (3) and (5) shall lie on (4).

If $\bar{x}_1$, $\bar{y}_1$ be the co-ordinates of the centre of (3), $\bar{x}_2$, $\bar{y}_2$ of (5), we have

$$\bar{x}_1 + kg' = 0, \quad \bar{x}_2 + kg' + 2\lambda\xi = 0,$$
$$k\bar{y}_1 + f = 0, \quad k\bar{y}_2 + f + 2\lambda\eta = 0.$$

Now, if $\bar{x}$, $\bar{y}$ be the co-ordinates of the point midway between these two, we have

$$\bar{x} = \tfrac{1}{2}(\bar{x}_1 + \bar{x}_2), \quad \bar{y} = \tfrac{1}{2}(\bar{y}_1 + \bar{y}_2).$$

Hence $\qquad \bar{x} + kg' + \lambda\xi = 0, \quad k\bar{y} + f + \lambda\eta = 0.$

And if $(\bar{x}, \bar{y})$ lie in (1), we have

$$\xi(kg' + \lambda\xi) + \frac{\eta}{k}(f + \lambda\eta) + 1 = 0.$$

Hence
$$\lambda = \frac{\xi kg' + \dfrac{\eta f}{k} + 1}{\xi^2 + \dfrac{\eta^2}{k}}.$$

The particular values of λ corresponding to (1) and (2) will be similarly found to be $\dfrac{f}{\eta}$, $\dfrac{g'}{\xi}$, or

$$\lambda_1 = \frac{f}{\eta}, \quad \lambda_2 = \frac{g'}{\xi};$$

therefore, by (6), $\dfrac{f}{\eta} + k\dfrac{g'}{\xi} = \dfrac{kg'\xi + \dfrac{\eta f}{k} + 1}{\xi^2 + \dfrac{\eta^2}{k}}$,

or $\qquad (f\xi + kg'\eta)\left(\xi^2 + \dfrac{\eta^2}{k}\right) = \left(kg'\xi + \dfrac{\eta f}{k} + 1\right)\xi\eta;$

therefore $\qquad f\xi^3 + g'\eta^3 - \xi\eta = 0 \dots\dots\dots\dots\dots\dots (7).$

This relation between ξ and η, being independent of k, proves that an infinite number of straight lines can be found, to satisfy the required condition.

The envelope is found by expressing the condition that two of the three roots of the equation

$$f\xi^3 + g'\eta^3 - \xi\eta(\xi x + \eta y) = 0,$$

considered as a cubic in $\xi : \eta$ shall be equal. This is

$$27f'^2g'^2 - 4(fy^3 + g'x^3) - 18fg'xy - x^2y^2 = 0 \ldots\ldots (8).$$

This investigation, though analytically complete in all cases, is perhaps open to exception on geometrical grounds, when the four points form a concave quadrangle. This want may be supplied as follows:

Every conic of the system will be an hyperbola, and one a rectangular hyperbola. Take, as co-ordinate axes, the lines through the centre of gravity of the four points parallel to its asymptote. Let its equation be

$$\frac{2xy}{ab} + \frac{2px}{a} + \frac{2qy}{b} + h = 0 \ldots\ldots\ldots\ldots\ldots (\alpha),$$

and let that of the hyperbola whose axes are parallel to its asymptotes, be

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{2qx}{a} + \frac{2py}{b} + h' = 0 \ldots\ldots\ldots\ldots (\beta).$$

(The coefficients of x and y in the one equation must be the same as those of $-y$ and x in the other, on account of the origin being the centre of gravity of the four points, that is, the centre of the conic which is the locus of the centres of the system.)

Multiplying equation (α) by $\pm\sqrt{(-1)}$ and adding it to (β), we get

$$\left\{\frac{x}{a} \pm \sqrt{(-1)}\,\frac{y}{b}\right\}^2 - 2\{q \mp \sqrt{(-1)}\,p\}\left\{\frac{x}{a} \mp \sqrt{(-1)}\,\frac{y}{b}\right\} + h' \pm \sqrt{(-1)}\,h = 0.$$

This equation represents the two imaginary parabolas which belong to the system. The envelope required may be seen, by modifying equation (8), to be

$$\begin{aligned}27(q^2+p^2)^2 + 4\Big[&\{q - \sqrt{(-1)}\,p\}\left\{\frac{x}{a} - \sqrt{(-1)}\,\frac{y}{b}\right\}^3 \\ &+ \{q + \sqrt{(-1)}\,p\}\left\{\frac{x}{a} + \sqrt{(-1)}\,\frac{y}{b}\right\}^3\Big] \\ &- 18(q^2+p^2)\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right) - \left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)^2 = 0,\end{aligned}$$

or

$$\begin{aligned}27(q^2+p^2)^2 + 8&\left\{q\left(\frac{x^3}{a^3} - \frac{3xy^2}{ab^2}\right) + p\left(\frac{y^3}{b^3} - \frac{3x^2y}{a^2b}\right)\right\} \\ &- 18(q^2+p^2)\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right) - \left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)^2 = 0.\end{aligned}$$

If $a=b$, in which case every conic becomes a rectangular hyperbola, and the locus of the centres a circle, of radius $(p^2+q^2)^{\frac{1}{2}}a$, this becomes

$$27(q^2+p^2)^2a^4+8a\{q(x^3-3xy^2)+p(y^3-3x^2y)\}$$
$$-18(q^2+p^2)a^2(x^2+y^2)-(x^2+y^2)^2=0 \ldots\ldots (9).$$

If we transform this to polar co-ordinates, and let $p=m\cos\alpha$, $q=m\sin\alpha$, it becomes

$$27m^4a^4+8mar^3\cos(3\theta-\alpha)-18m^2a^2r^2-r^4=0 \ldots (10).$$

We will next proceed to investigate the locus of the four points in which the variable group of conics intersects. For this purpose, returning to the original form, take the two parabolas of the group, whose equations are

$$S+\frac{f}{\eta}(\xi x+\eta y-1)=0,$$

$$S'+\frac{g'}{\xi}(\xi x+\eta y-1)=0.$$

From this equation we get

$$\frac{\xi}{g'S}=\frac{\eta}{fS'}=\frac{1}{SS'+g'Sx+fS'y},$$

and by (7) we get for the locus

$$fg'^3S^3+g'f^3S'^3-fg'SS'(SS'+g'Sx+fS'y),$$

or
$$g'^2S^3+f^2S'^3-SS'(SS'+g'Sx+fS'y)=0.$$

The left-hand member of this equation is of the eighth degree. It breaks up however into four factors of the second degree, as may be proved as follows. If in it we put $S=0$, we get $S'^3=0$, and *vice versâ*, proving that each of the four given points is a triple point on the curve. Again, the terms of the highest degree in x and y reduce simply to x^4y^4, proving that each of the points where the line at infinity meets the co-ordinate axes (*i.e.* the two points at infinity on the locus of centres) is a quadruple point on the curve. Now, any conic section in general meets a curve of the eighth degree in 2×8 or 16 points. If however we draw a conic through three of the four given points, and the two points where the line at infinity meets the co-ordinate axes, this conic will meet the curve in what is equivalent to $3\times 3+2\times 4$ or 17 points. Hence it must coincide with the curve throughout the whole length of the conic, or the conic forms a part of

the curve. Similarly, each of the conics through every three of the four given points and the two points at infinity, forms a part of the curve. Hence, the locus is made up of the four conics, similar and similarly situated to the locus of centres, which pass through each three of the four given points.

In the particular case in which the line joining each pair of points is perpendicular to the line joining the other pair, these curves become the four circles passing through each three of the given points; and it may be seen, by regarding the three pairs of straight lines drawn through the four points as curves of the group (3), that the system of straight lines $\xi x + \eta y - 1 = 0$, becomes the line joining the feet of the perpendiculars let fall from any point in the circumference of the circumscribing circle. The envelope of this system is represented by the equation (9) or (10); either of which represents the three-cusped hypocycloid, as may be shown as follows. Writing in equation (10), c for ma, let c be the radius of the rolling, $3c$ of the fixed circle, and take as axis of x the line joining the centre of the fixed circle with the initial point of contact of the circles. Then, when the centre of the rolling circle has described an arc $2c\phi$, we have

$$x = 2c\cos\phi + c\cos 2\phi,$$

$$y = 2c\sin\phi - c\sin 2\phi;$$

therefore $x + \sqrt{(-1)}\,y = r\varepsilon^{\theta\sqrt{(-1)}} = 2c\varepsilon^{-\phi\sqrt{(-1)}} + c\varepsilon^{-2\phi\sqrt{(-1)}},$

$$r\varepsilon^{-\theta\sqrt{(-1)}} = 2c\varepsilon^{-\phi\sqrt{(-1)}} + c\varepsilon^{2\phi\sqrt{(-1)}};$$

therefore $\qquad r^2 = 5c^2 + 4c^2\cos 3\phi.$

And $\quad 2r^3\cos 3\theta = (r\varepsilon^{\theta\sqrt{(-1)}})^3 + (r\varepsilon^{-\theta\sqrt{(-1)}})^3$

$$= 16c^3\cos 3\phi + 24c^3 + 12c^3\cos 3\phi + 2c^3\cos 6\phi,$$

or $\qquad r^3\cos 3\theta = c^3(12 + 14\cos 3\phi + \cos 6\phi).$

Also $\quad (r^2 + 9c^2)^2 = c^4(14 + 4\cos 3\phi)^2$

$$= c^4(196 + 112\cos 3\phi + 16\cos^2 3\phi)$$

$$= c^4(204 + 112\cos 3\phi + 8\cos 6\phi);$$

therefore $\qquad (r^2 + qc^2)^2 - 8cr^3\cos 3\theta = 108c^4,$

or $\qquad r^4 + 18c^2r^2 - 8cr^3\cos 3\theta - 27c^4 = 0,$

which is identical with equation (10).

This result is investigated independently in a paper printed in the present number of the *Journal.*

August 10, 1866.

ON A LOCUS IN RELATION TO THE TRIANGLE.

By Professor CAYLEY.

IF from any point of a circle circumscribed about a triangle perpendiculars are let fall upon the sides, the feet of the perpendiculars lie in a line; or, what is the same thing, the locus of a point, such that the perpendiculars let fall therefrom upon the sides of a given triangle have their feet in a line, is the circle circumscribed about the triangle.

In this well known theorem we may of course replace the circular points at infinity by any two points whatever; or the Absolute being a point-pair, and the terms perpendicular and circle being understood accordingly, we have the more general theorem expressed in the same words.

But it is less easy to see what the corresponding theorem is, when instead of being a point-pair, the Absolute is a proper conic; and the discussion of the question affords some interesting results.

Take $(x=0, y=0, z=0)$ for the equations of the sides of the triangle, and let the equation of the Absolute be

$$(a, b, c, f, g, h \between x, y, z)^2 = 0,$$

then any two lines which are harmonics in regard to this conic (or, what is the same thing, which are such that the one of them passes through the pole of the other) are said to be perpendicular to each other, and the question is—

Find the locus of a point, such that the perpendiculars let fall therefrom on the sides of the triangle have their feet in a line.

Supposing, as usual, that the inverse coefficients are (A, B, C, F, G, H), and that K is the discriminant, the co-ordinates of the poles of the three sides respectively are (A, H, G), (H, B, F), (G, F, C). Hence considering a point P, the co-ordinates of which are (x, y, z), and taking (X, Y, Z) for current co-ordinates, the equation of the perpendicular from P on the side $X=0$ is

$$\begin{vmatrix} X, & Y, & Z \\ x, & y, & z \\ A, & H, & G \end{vmatrix} = 0,$$

and writing in this equation $X=0$, we find

$$(0,\ Ay - Hx,\ Az - Gx)$$

for the co-ordinates of the foot of the perpendicular. For the other perpendiculars respectively, the co-ordinates are

$$(Bx - Hy,\quad 0,\quad Bz - Fy),$$

and $$(Cx - Gz,\ Cy - Fz,\quad 0\quad),$$

and hence the condition in order that the three feet may lie in a line is

$$\begin{vmatrix} 0 & , Ay - Hx, & Az - Gx \\ Bx - Hy, & 0 & , Bz - Fy \\ Cx - Gz, & Cy - Fz, & 0 \end{vmatrix} = 0;$$

or, what is the same thing,

$$\begin{aligned}&(Ay - Hx)(Bz - Fy)(Cx - Gz)\\ &+ (Az - Gx)(Bx - Hy)(Cy - Fz) = 0,\end{aligned}$$

that is
$$\begin{aligned}&2(ABC - FGH)\,xyz\\ &+ A(\ FH - \ BG)\,yz^2\\ &+ A(\ FG - \ CH)\,y^2z\\ &+ B(\ FG - \ CH)\,zx^2\\ &+ B(\ GH - \ AF)\,z^2x\\ &+ C(\ GH - \ AF)\,xy^2\\ &+ C(\ HF - \ BG)\,x^2y = 0,\end{aligned}$$

which is the equation of the locus of P; the locus is therefore a cubic. Writing for a moment

$$\begin{aligned}(BC-F^2,\ CA-G^2,\ AB-H^2,\ GH-AF,\ HF-BG,\ FG-CH)\\ = (A',\ B',\ C',\ F',\ G',\ H'),\end{aligned}$$

and K' for the discriminant $ABC - AF^2 -$&c., the equation is

$$\begin{aligned}&2(ABC - FGH)\,xyz\\ &+ Ayz(H'y + G'z) + Bzx(H'x + F'z) + Cxy(G'x + F'y) = 0,\end{aligned}$$

or as this may also be written

$$\begin{aligned}&\frac{2}{F'G'H'}(ABC - FGH)\,xyz\\ &+ \frac{A}{F'}\,yz\left(\frac{y}{G'} + \frac{z}{H'}\right) + \frac{B}{G'}\,zx\left(\frac{x}{F'} + \frac{z}{H'}\right) + \frac{C}{H'}\,xy\left(\frac{x}{F'} + \frac{y}{G'}\right) = 0,\end{aligned}$$

that is $$\left[\frac{2}{F'G'H'}(ABC - FGH) - \frac{A}{F'^2} - \frac{B}{G'^2} - \frac{C}{H'^2}\right] xyz$$

$$+ \left(\frac{A}{F'}\,yz + \frac{B}{G'}\,zx + \frac{C}{H'}\,xy\right)\left(\frac{x}{F'} + \frac{y}{G'} + \frac{z}{H'}\right) = 0,$$

and the cubic will therefore break up into a line and conic if only

$$\frac{2}{F'G'H'}(ABC-FGH)-\frac{A}{F'^2}-\frac{B}{G'^2}-\frac{C}{H'^2}=0,$$

and it is easy to see that conversely this is the necessary and sufficient condition in order that the cubic may so break up.

The condition is

$$\Omega=2F'G'H'(ABC-FGH)-AG'^2H'^2-BH'^2F'^2-CF'^2G'^2=0,$$

we have $AA'+BB'+CC'=3ABC-AF^2-BG^2-CH^2$

$$=K'+2(ABC-FGH),$$

and thence $\Omega=F'G'H'(AA'+BB'+CC'-K')$

$$-AG'^2H'^2-BH'^2F'^2-CF'^2G'^2,$$

that is

$$\Omega=-AG'H'(G'H'-A'F')-BH'F'(H'F'-B'G')$$
$$-CF'G'(F'G'-C'H')-K'F'G'H'$$
$$=-AG'H'K'F-BH'F'K'G-CF'G'K'H-K'F'G'H'$$
$$=-K'(AFG'H'+BGH'F'+CHF'G'+F'G'H'),$$

so that the condition $\Omega=0$ is satisfied if $K'=0$, that is if the equation

$$(A, B, C, F, G, H)(\xi, \eta, \zeta)^2=0,$$

which is the line-equation of the Absolute breaks up into factors; that is, if the Absolute be a point-pair.

In the case in question we may write

$$(A, B, C, F, G, H)(\xi, \eta, \zeta)^2=2(\alpha\xi+\beta\eta+\gamma\zeta)(\alpha'\xi+\beta'\eta+\gamma'\zeta),$$

that is

$$(A,B,C,F,G,H)=(2\alpha\alpha', 2\beta\beta', 2\gamma\gamma', \beta\gamma'+\beta'\gamma, \gamma\alpha'+\gamma'\alpha, \alpha\beta'+\alpha'\beta),$$

whence also, putting for shortness,

$$(\beta\gamma'-\beta'\gamma, \gamma\alpha'-\gamma'\alpha, \alpha\beta'-\alpha'\beta)=(\lambda, \mu, \nu),$$

we have

$$(A', B', C', F', G', H')=-(\lambda^2, \mu^2, \nu^2, \mu\nu, \nu\lambda, \lambda\mu),$$

and also $K'=0$, $2(ABC-FGH)=AA'+BB'+CC'$

$$=-2(\alpha\alpha'\lambda^2+\beta\beta'\mu^2+\gamma\gamma'\nu^2).$$

The original cubic equation is

$$(\alpha\alpha'\lambda^2+\beta\beta'\mu^2+\gamma\gamma'\nu^2)\,xyz$$
$$+\alpha\alpha'\lambda yz(\mu y+\nu z)+\beta\beta'\mu zx(\lambda x+\nu z)+\gamma\gamma'\nu xy(\lambda x+\mu y)=0,$$

and this in fact is

$$(\alpha\alpha'\lambda yz + \beta\beta'\mu zx + \gamma\gamma'\nu xy)(\lambda x + \mu y + \nu z) = 0.$$

The equation $\lambda x + \mu y + \nu z = 0$ is in fact that of the line through the two points which constitute the Absolute; the other factor gives

$$\alpha\alpha'\lambda yz + \beta\beta'\mu zx + \gamma\gamma'\nu xy = 0,$$

which is the equation of a conic through the angles of the triangle $(x = 0,\ y = 0,\ z = 0)$, and which also passes through the two points of the Absolute; in fact, writing (α, β, γ) for (x, y, z) the equation becomes $\alpha\beta\gamma(\alpha'\lambda + \beta'\mu + \gamma'\nu) = 0$, and so also writing $(\alpha', \beta', \gamma')$ for (x, y, z) it becomes $\alpha'\beta'\gamma'(\alpha\lambda + \beta\mu + \gamma\nu) = 0$, which relations are identically satisfied by the values of (λ, μ, ν). Hence we see that the Absolute being a point-pair, the locus is the conic passing through the angles of the triangle, and the two points of the Absolute; that is, it is the *circle* passing through the angles of the triangle.

But assuming that K' is not $= 0$, or that the Absolute is a proper conic, the equation $\Omega = 0$ will be satisfied if

$$AFG'H' + BGH'F' + CHF'G' + F'G'H' = 0,$$

we have $F', G', H' = Kf, Kg, Kh$ respectively, or omitting the factor K^2, the equation becomes

$$AFgh + BGhf + CHfg + Kfgh = 0,$$

which in fact is

$$f^2g^2h^2 - bcg^2h^2 - cah^2f^2 - abf^2g^2 + 2abcfgh = 0,$$

or, as it may also be written,

$$abcf^2g^2h^2\left(\frac{1}{abc} - \frac{1}{af^2} - \frac{1}{bg^2} - \frac{1}{ch^2} + \frac{2}{fgh}\right) = 0.$$

I remark that we have $ABC - FGH = K(abc - fgh)$; substituting also for F', G', H' the values Kf, Kg, Kh, the equation of the cubic curve is

$$2(abc - fgh)xyz$$
$$+ Ayz(hy + gz) + Bzx(hx + fy) + Cxy(gx + fy) = 0,$$

and the transformed form is

$$\left[\frac{2}{fgh}(abc - fgh) - \frac{A}{f^2} - \frac{B}{g^2} - \frac{C}{h^2}\right]xyz$$
$$+ \left(\frac{A}{f}yz + \frac{B}{g}zx + \frac{C}{h}xy\right)\left(\frac{x}{f} + \frac{y}{g} + \frac{z}{h}\right) = 0,$$

and we in fact have

$$\frac{2}{fgh}(abc-fgh)-\frac{A}{f^2}-\frac{B}{g^2}-\frac{C}{h^2}$$
$$=abc\left(\frac{1}{abc}-\frac{1}{af^2}-\frac{1}{bg^2}-\frac{1}{ch^2}+\frac{2}{fgh}\right),$$

so that the foregoing condition

$$\frac{1}{abc}-\frac{1}{af^2}-\frac{1}{bg^2}-\frac{1}{ch^2}+\frac{2}{fgh}=0,$$

being satisfied, the cubic breaks up into the line $\frac{x}{f}+\frac{y}{g}+\frac{z}{h}=0$, and the conic

$$\frac{A}{f}yz+\frac{B}{g}zx+\frac{C}{h}xy=0.$$

It is to be remarked that in general a triangle and the reciprocal triangle are in perspective; that is, the lines joining corresponding angles meet in a point, and the points of intersections of opposite sides lie in a line; this is the case therefore with the triangle $(x=0,\ y=0,\ z=0)$, and the reciprocal triangle

$$(ax+hy+gz=0,\ \ hx+by+fz=0,\ \ gx+fy+cz=0);$$

and it is easy to see that the line through the points of intersection of corresponding sides is in fact the above mentioned line $\frac{x}{f}+\frac{y}{g}+\frac{z}{h}=0$. It is to be noticed also that the coordinates of the point of intersection of the lines joining the corresponding angles are (F, G, H). The conic

$$\frac{A}{f}yz+\frac{B}{g}zx+\frac{C}{h}xy=0$$

is of course a conic passing through the angles of the triangle $(x=0,\ y=0,\ z=0)$; it is *not*, what it might have been expected to be, a conic having double contact with the Absolute $(a, b, c, f, g, h)(x, y, z)^2$.

I return to the condition

$$\frac{1}{abc}-\frac{1}{af^2}-\frac{1}{bg^2}-\frac{1}{ch^2}+\frac{2}{fgh}=0,$$

this can be shown to be the condition in order that the sides of the triangle $(x=0,\ y=0,\ z=0)$, and the sides of the reciprocal

triangle $(ax+hy+gz=0,\ hx+by+fz=0,\ gx+fy+cz=0)$ touch one and the same conic; in fact, using line coordinates, the coordinates of the first three sides are (1, 0, 0), (0, 1, 0), (0, 0, 1) respectively, and those of the second three sides are (a, h, g), (h, b, f), (g, f, c) respectively; the equation of a conic touching the first three lines is

$$\frac{L}{\xi}+\frac{M}{\eta}+\frac{N}{\zeta}=0,$$

and hence making the conic touch the second three sides, we have three linear equations from which eliminating L, M, N, we find

$$\begin{vmatrix} \frac{1}{a}, & \frac{1}{h}, & \frac{1}{g} \\ \frac{1}{h}, & \frac{1}{b}, & \frac{1}{f} \\ \frac{1}{g}, & \frac{1}{f}, & \frac{1}{c} \end{vmatrix}=0,$$

which is the equation in question.

We know that if the sides of two triangles touch one and the same conic, their angles must lie in and on the same conic. The co-ordinates of the angles are (1, 0, 0), (0, 1, 0), (0, 0, 1) and (A, H, G), (H, B, F), (G, F, C) respectively, and the angles will be situate in a conic if only

$$\begin{vmatrix} \frac{1}{A}, & \frac{1}{H}, & \frac{1}{G} \\ \frac{1}{H}, & \frac{1}{B}, & \frac{1}{F} \\ \frac{1}{G}, & \frac{1}{F}, & \frac{1}{C} \end{vmatrix}=0,$$

an equation which must be equivalent to the last preceding one; this is easily verified. In fact, writing for shortness

$$\Delta=\begin{vmatrix} \frac{1}{a}, & \frac{1}{h}, & \frac{1}{g} \\ \frac{1}{h}, & \frac{1}{b}, & \frac{1}{f} \\ \frac{1}{g}, & \frac{1}{f}, & \frac{1}{c} \end{vmatrix},\quad \square=\begin{vmatrix} \frac{1}{A}, & \frac{1}{H}, & \frac{1}{G} \\ \frac{1}{H}, & \frac{1}{B}, & \frac{1}{F} \\ \frac{1}{G}, & \frac{1}{F}, & \frac{1}{C} \end{vmatrix},$$

we have $$-\square = \frac{1}{ABCF^2}(BC - F^2)$$

$$+ \frac{1}{CFGH^2}(FG - CH) + \frac{1}{BFG^2H}(HF - BG)$$

$$= \frac{K}{ABCF^2G^2H^2}(aG^2H^2 + hABFG + gCAHF),$$

and the second factor is

$$= aGH(AF + Kf) + AFhBG + AFgCH$$
$$= AF(aGH + hBG + gCH) + KafGH.$$

But

$$\begin{aligned} aGH + hBG + gCH &= G(aH + hB) + gCH = G - gF + gCH \\ &= G - gF + gCH \\ &= -g(FG - CH) \\ &= -ghK, \end{aligned}$$

so that the second factor is

$$= K(afGH - ghAF),$$

which is

$$= K(f^2g^2h^2 - bcg^2h^2 - cah^2f^2 - abf^2g^2 + 2abcfgh)$$
$$= Kabcf^2g^2h^2\left(\frac{1}{abc} - \frac{1}{af^2} - \frac{1}{bg^2} - \frac{1}{ch^2} + \frac{2}{fgh}\right)$$
$$= Kabcf^2g^2h^2\nabla,$$

so that we have identically

$$-ABCF^2G^2H^2\,\square = K^2abcf^2g^2h^2\nabla,$$

and the conditions $\nabla = 0$, $\square = 0$ are consequently equivalent.

The condition

$$\frac{1}{abc} - \frac{1}{af^2} - \frac{1}{bg^2} - \frac{1}{ch^2} + \frac{2}{fgh} = 0,$$

is the condition in order that the function

$$\left(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}, \frac{1}{f}, \frac{1}{g}, \frac{1}{h}\right)(ax, by, cz)^2,$$

may break up into linear factors; the function in question is

$$\left(a, b, c, \frac{bc}{f}, \frac{ca}{g}, \frac{ab}{h}\right)(x, y, z)^2,$$

which is

$$= (a, b, c, f, g, h\between x, y, z)^2 + 2\left(\frac{A}{f}yz + \frac{B}{g}zx + \frac{C}{h}xy\right),$$

so that the condition is, that the conic

$$(a, b, c, f, g, h\between x, y, z)^2 + 2\left(\frac{A}{f}yz + \frac{B}{g}zx + \frac{C}{h}xy\right) = 0,$$

(which is a certain conic passing through the intersections of the Absolute $(a, b, c, f, g, h\between x, y, z)^2 = 0$, and of the locus conic $\frac{A}{f}yz + \frac{B}{g}zx + \frac{C}{h}xy = 0$) shall be a pair of lines. Writing the equation of the conic in question under the form

$$\left(a, b, c, \frac{bc}{f}, \frac{ca}{g}, \frac{ab}{h}\between x, y, z\right)^2 = 0,$$

the inverse coefficients A', B', C', F', G', H' of this conic, are

$$\left(-\frac{Abc}{f^2}, -\frac{Bca}{g^2}, -\frac{Cab}{h^2}, -\frac{abc}{fgh}F, -\frac{abc}{fgh}G, -\frac{abc}{fgh}H\right),$$

so that we have $F' : G' : H' = F : G : H$. Hence, if in regard to this new conic we form the reciprocal of the triangle $(x=0, y=0, z=0)$, and join the corresponding angles of the two triangles, the joining lines meet in a point which is the same point as is obtained by the like process from the triangle and its reciprocal in regard to the Absolute. But I do not further pursue this part of the theory.

It is to be noticed that the conic

$$\frac{A}{f}yz + \frac{B}{g}zx + \frac{C}{h}xy = 0,$$

contains the angles of the reciprocal triangle, and is thus in fact the conic in which are situate the angles of the two triangles. For the co-ordinates of one of the angles of the reciprocal triangle are (A, H, G); we should therefore have

$$\frac{A}{f}HG + \frac{B}{g}GA + \frac{C}{h}AH = 0,$$

which is $$\frac{A}{fgh}(GHgh + BGhf + CHfg) = 0,$$

or attending only to the second factor and writing

$$GH = Kf + AF,$$

the condition is

$$Kfgh + AFgh + BGhf + CHfg = 0,$$

or substituting for K, A, B, C, F, G, H their values and reducing, this is

$$-abcf^2g^2h^2\left(\frac{1}{abc} - \frac{1}{af^2} - \frac{1}{bg^2} - \frac{1}{ch^2} + \frac{2}{fgh}\right) = 0,$$

which is satisfied: hence the three angles of the reciprocal triangle lie on the conic in question.

Partially recapitulating the foregoing results, we see in the case where the Absolute is not a point-pair, that the locus of a point such that the perpendiculars from it on the sides of the triangle have their feet in a line, is in general a *cubic curve* passing through the angles of the triangle: if, however, the condition

$$\frac{1}{abc} - \frac{1}{af^2} - \frac{1}{bg^2} - \frac{1}{ch^2} + \frac{2}{fgh} = 0$$

be satisfied, that is, if the triangle be such that the angles thereof and of the reciprocal triangle lie in a conic (or, what is the same thing, if the sides touch a conic) then the cubic locus breaks up into the line $\frac{x}{f} + \frac{y}{g} + \frac{z}{h} = 0$, which is the line through the points of intersection of the corresponding sides of the two triangles, and into the conic

$$\frac{A}{f}yz + \frac{B}{g}zx + \frac{C}{h}xy = 0,$$

which is the conic through the angles of the two triangles.

The question arises, given a conic (the Absolute) to construct a triangle such that its angles, and the angles of the reciprocal triangle in regard to the given conic, lie in a conic.

I suppose that two of the angles of the triangle are given, and I enquire into the locus of the remaining angle. To fix the ideas, let A, B, C be the angles of the triangle, A', B', C' those of the reciprocal triangle; and let the angles A and B be given. We have to find the locus of the point C: I observe however, that the lines AA', BB', CC' meet in a point O, and I conduct the investigation in such manner as to obtain simultaneously the loci of the two points C and O. The lines $C'B'$, $C'A'$ are the polars of A, B respectively, let their equations be $x = 0$, and $y = 0$, and

let the equation of the line AB be $z=0$; this being so, the equation of the given conic will be of the form

$$(a, b, c, 0, 0, h \between x, y, z)^2 = 0.$$

I take (α, β, γ) for the co-ordinates of O and (x, y, z) for those of C; the co-ordinates of either of these points being of course deducible from those of the other.

Observing that the inverse coefficients are

$$(bc, ca, ab-h^2, 0, 0, -ch),$$

we find co-ordinates of A are $(\ \ b, -h, 0)$,

" B " $(-h, \ \ a, 0)$.

The points A' and B' are then given as the intersections of AO with $C'A'$ $(y=0)$ and of BO with $C'B'$ $(x=0)$; we find

co-ordinates of A' are $(h\alpha+b\beta, \quad 0 \quad, h\gamma)$,

" B' " $(\quad 0 \quad, a\alpha+h\beta, h\gamma)$.

Moreover, co-ordinates of C' are $(0, 0, 1)$,

" C " (x, y, z).

The six points A, B, C, A', B', C' are to lie in a conic; the equations of the lines $C'A$, $C'B$, AB are $hX+bY=0$, $aX+hY=0$, $Z=0$, and hence the equation of a conic passing through the points C', A, B is

$$\frac{L}{aX+hY}+\frac{M}{hX+bY}+\frac{N}{Z}=0.$$

Hence, making the conic pass through the remaining points A', B', C, we find

$$\frac{L}{a(h\alpha+b\beta)}+\frac{M}{h(h\alpha+b\beta)}+\frac{N}{h\gamma}=0,$$

$$\frac{L}{h(a\alpha+h\beta)}+\frac{M}{b(a\alpha+h\beta)}+\frac{N}{h\gamma}=0,$$

$$\frac{L}{ax+hy}+\frac{M}{hx+by}+\frac{N}{z}=0,$$

and eliminating the L, M, N, we find

$$\begin{vmatrix} \frac{1}{a}, & \frac{1}{h}, & h\alpha+b\beta \\ \frac{1}{h}, & \frac{1}{b}, & a\alpha+h\beta \\ \frac{1}{ax+hy}, & \frac{1}{hx+by}, & \frac{hy}{z} \end{vmatrix}=0,$$

or developing and reducing, this is

$$-\frac{(ab-h^2)}{hab}\frac{y}{z}+\frac{1}{h}\frac{a\alpha+h\beta}{ax+hy}+\frac{1}{h}\frac{h\alpha+b\beta}{hx+by}$$
$$-\frac{1}{a}\frac{a\alpha+h\beta}{hx+by}-\frac{1}{b}\frac{h\alpha+b\beta}{ax+hy}=0.$$

We have still to find the relation between (α, β, γ) and (x, y, z); this is obtained by the consideration that the line $A'B'$, through the two points A', B' the co-ordinates of which are known in terms of (α, β, γ), is the polar of the point C, the co-ordinates of which are (x, y, z). The equation of $A'B'$ is thus obtained in the two forms

$$(a\alpha+h\beta)X+(h\alpha+b\beta)Y-\frac{(a\alpha+h\beta)(h\alpha+b\beta)}{hy}Z=0,$$

and $(ax+hy)X+(hx+by)Z+ \qquad cz \qquad Z=0,$

and comparing these, we have

$$x:y:z=\alpha:\beta:\frac{-(a\alpha+h\beta)(h\alpha+b\beta)}{chy},$$

or what is the same thing

$$\alpha:\beta:\gamma=x:y:\frac{-(ax+hy)(hx+by)}{chz},$$

(where it is to be observed that the equation $\alpha:\beta=x:y$ is the verification of the theorem that the lines AA', BB', CC' meet in a point O).

We may now from the above found relation eliminate either the (α, β, γ) or the (x, y, z); first eliminating the (α, β, γ), we find

$$-\frac{ab-h^2}{hab}\frac{Y}{Z}+\frac{2}{h}-\frac{1}{a}\frac{ax+hy}{hx+by}-\frac{1}{b}\frac{hx+by}{ax+by}=0,$$

where $$\frac{Y}{Z}=-\frac{(ax+hy)(hx+by)}{chz^2},$$

or completing the elimination

$$\frac{ab-h^2}{ch}\frac{(ax+hy)^2(hx+by)^2}{z^2}=(hb,\ -ab,\ ha\between ax+hy,\ hx+by)^2=0,$$

which is a quartic curve having a node at each of the points

$(z=0,\ ax+hy=0)$, $(z=0,\ hx+by=0)$, $(ax+hy=0,\ hx+by=0)$,

that is, at each of the points B, A, C'. The right-hand side of the foregoing equation is

$$=-(ab-h^2)(ha,\ ab,\ hb)(x,\ y)^2, =-(ab-h^2)\,h\left(ax^2+by^2+\frac{2ab}{h}\,xy\right),$$

so that the equation may also be written

$$(ax+hy)^2\,(hx+by)^2+ch^2z^2\left(ax^2+by^2+\frac{2ab}{h}\,xy\right)=0.$$

Secondly, to eliminate the (x, y, z), we have

$$-\frac{ab-h^2}{hab}\,\frac{Y}{Z}+\frac{2}{h}-\frac{1}{a}\,\frac{a\alpha+h\beta}{h\alpha+b\beta}-\frac{1}{b}\,\frac{h\alpha+b\beta}{a\alpha+h\beta}=0,$$

where $$\frac{Y}{Z}=-\frac{chy^2}{(a\alpha+h\beta)\,(h\alpha+b\beta)},$$

or completing the elimination

$$(ab-h^2)\,chy^2=\quad(hb,\ -ab,\ ha)(a\alpha+h\beta,\ h\alpha+b\beta)^2$$
$$=-(ab-h^2)\,h\left(a\alpha^2+b\beta^2+\frac{2ab}{h}\,\alpha\beta\right),$$

that is $$\left(a,\ b,\ c,\ 0,\ 0,\ \frac{ab}{h}\right)(\alpha,\ \beta,\ \gamma)^2=0,$$

Or writing (x, y, z) in place of (α, β, γ), the locus of the point O is the conic

$$\left(a,\ b,\ c,\ 0,\ 0,\ \frac{ab}{h}\right)(x,\ y,\ z)^2=0,$$

which is a conic intersecting the Absolute

$$(a,\ b,\ c,\ 0,\ 0,\ h)(x,\ y,\ z)^2=0,$$

at its intersections with the lines $x=0$, $y=0$, that is the lines $C'B'$ and $C'A'$.

In regard to this new conic, the co-ordinates of the pole of $C'B'$ $(x=0)$ are at once found to be $(-h, a, 0)$, that is, the pole of $C'B'$ is B; and similarly the co-ordinates of the pole of $C'A'$ $(y=0)$ are $(b, -h, 0)$, that is, the pole of $C'A'$ is A. We may consequently construct the conic the locus of O, viz. given the Absolute and the points A and B, we have $C'A'$ the polar of B, meeting the Absolute in two points (a_1, a_2), and $C'B'$ the polar of A meeting the Absolute in the points (b_1 and b_2); the lines $C'A'$ and $C'B'$ meet in C'. This being so, the required conic passes through the points a_1, a_2, b_1, b_2, the tangents at these points being Aa_1, Aa_2, Bb_1, Bb_2 respectively, eight conditions, five of which would

be sufficient to determine the conic. It is to be remarked that the lines $C'B'$, $C'A'$ (which in regard to the Absolute are the polars of A, B respectively) are in regard to the required conic the polars of B, A respectively.

The conic the locus of O being known, the point O may be taken at any point of this conic, and then we have A' as the intersection of $C'A'$ with AO, B' as the intersection of $C'B'$ with BO, and finally, C as the pole of the line $A'B'$ in regard to the Absolute, the point so obtained being a point on the line $C'O$. To each position of O on the conic locus, there corresponds of course a position of C, the locus of C is, as has been shown, a quartic curve having a node at each of the points C', A, B.

The foregoing conclusions apply of course to spherical figures; we see therefore that on the sphere the locus of a point such that the perpendiculars let fall on the sides of a given spherical triangle have their feet in a line (great circle), is a spherical cubic. If, however, the spherical triangle is such that the angles thereof and the poles of the sides (or, what is the same thing, the angles of the polar triangle) lie on a spherical conic; then the cubic locus breaks up into a line (great circle), which is in fact the circle having for its pole the point of intersection of the perpendiculars from the angles of the triangle on the opposite sides respectively, and into the before-mentioned spherical conic. Assuming that the angles A and B are given, the above-mentioned construction, by means of the point O, is applicable to the determination of the locus of the remaining angle C, in order that the spherical triangle ABC may be such that the angles and the poles of the sides lie on the same spherical conic, but this required some further developments. The lines $C'B'$, $C'A'$ which are the polars of the given angles A, B respectively, are the cyclic arcs of the conic the locus of O, or say for shortness the conic O; and moreover these same lines $C'B'$, $C'A'$ are in regard to the conic O, the polars of the angles B, A respectively. If instead of the conic O we consider the polar conic O', it follows that A, B are the foci, and $C'A'$, $C'B'$ the corresponding directrices of the conic O'. The distance of the directrix $C'A'$ from the centre of the conic, measuring such distance along the transverse axis is clearly $= 90° -$ distance of the focus A; it follows that the transverse semi-axis is $= 45°$, or what is the same thing, that the transverse axis is $= 90°$; that is, the conic O' is a conic described about the foci A, B with a transverse axis (or sum or difference of the focal distances) $= 90°$. Considering any

tangent whatever of this conic, the pole of the tangent is a position of the point O, which is the point of intersection of the perpendiculars let fall from the angles of the spherical triangle on the opposite sides; hence, to complete the construction, we have only through A and B respectively to draw lines AC, BC perpendicular to the lines BO, CO respectively; the lines in question will meet in a point C, which is such that CO will be perpendicular to AB, and which point C is the required third angle of the spherical triangle ABC. In order to ascertain whether a given spherical triangle ABC has the property in question (viz. whether it is such that the angles thereof and of the polar triangle lie in a spherical conic), we have only to construct as before the conic O' with the foci A, B and transverse axis $= 90°$, and then ascertain whether the polar of the point O, the intersections of the perpendiculars from the angles of the triangle on the opposite sides respectively, is a tangent of the conic O'. It is moreover clear, that given a triangle ABC having the property in question, then if with the foci A, B and transverse axis $= 90°$ we describe a conic, and if in like manner with the foci A, C and the same transverse axis, and with the foci B, C and same transverse axis, we describe two other conics, then that the three conics will having a common tangent the pole whereof will be the point of intersection of the perpendiculars from the angles of the triangle ABC on the opposite sides respectively.

TO THE EDITORS OF THE QUARTERLY JOURNAL OF PURE AND APPLIED MATHEMATICS.

GENTLEMEN,

Mr. Warren has pointed out to me that an equation given by me (*Higher Algebra*, Second Edition, p. 180, Art. 214, last line) connecting the Resultant and other Combinants of a system of two quartics, had been previously given by him (*Quarterly Journal*, Vol. VII., p. 70). The reason why I happened to overlook Mr. Warren's paper was that I wrote in the country where I had not many books with me, and Vol. VI. was the latest volume of the *Quarterly Journal* I had got.

GEO. SALMON.

Trinity College, Dublin,
Dec. 5, 1866.

ON AN INTEGRAL EXPRESSING THE "RANGE" OF CONDITIONED VARIABLES.

By C. J. Monro.

It is directly from geometrical construction that we usually deduce the integrals which express, by means of rectangular co-ordinates, the lengths and areas of lines and surfaces. But considering, first, that lines and surfaces are adequately defined by three variables conditioned by two equations and one equation respectively; and secondly, that rectangularity is adequately expressed by symmetrical algebraical formulas, it is evident that these integrals are comprehended in a more general integral, which shall involve any number of variables, and shall have its form determined by any not greater number of equations conditioning them, and by symmetrical formulas exactly like those expressing rectangularity. The general integral may be said, in a phrase due to Professor Donkin, to express the *range* of the variables, just as the particular integral expresses the range of a point upon a line or surface.

If a line or surface be defined by means of rectangular axes, of which one or two, as the case may be, touch the curve or surface at the origin, then will the element of length or area about that point be measured by a single differential or the product of two; and we shall only have to transform this expression in the most general way to other rectangular axes, and then integrate; after which it is a matter of elimination to express the result in terms of partial differential coefficients. Now after transformation our equations differentiated equate to zero the differentials or differential measured not along the line or surface, but perpendicularly to the surfaces or surface immediately defined by the equations or equation. In the case therefore of a line, the conditions of rectangularity are not satisfied unless the surfaces defining it happen to cut each other orthogonally: but we can always substitute two co-ordinate axes which will satisfy them.

Now suppose we had σ variables,

$$x_1,\ x_2,\ \ldots\ x_\sigma,$$

conditioned by τ equations of which the differentials were

$$\left.\begin{array}{l} U_{1.1}dx_1 + U_{1.2}dx_2 + \ldots + U_{1.\sigma}\,dx_\sigma = 0 \\ U_{2.1}\,dx_1 + U_{2.2}dx_2 + \ldots + U_{2.\sigma}\,dx_\sigma = 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ U_{\tau.1}dx_1 + U_{\tau.2}dx_2 + \ldots + U_{\tau.\sigma}dx_\sigma = 0 \end{array}\right\} \ldots\ldots\ldots (1),$$

and that making $\rho = \sigma - \tau$ we took ρ other variables defined by equations of which the differentials were

$$\left.\begin{array}{l} dv_1 = V_{1.1}dx_1 + V_{1.2}dx_2 + \ldots + V_{1.\sigma}dx_\sigma \\ dv_2 = V_{2.1}dx_1 + V_{2.2}dx_2 + \ldots + V_{2.\sigma}dx_\sigma \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ dv_\rho = V_{\rho.1}dx_1 + V_{\rho.2}dx_2 + \ldots + V_{\rho.\sigma}dx_\sigma \end{array}\right\} \ldots\ldots\ldots (2),$$

the coefficients in the σ equations being subject to $\frac{\sigma.\sigma+1}{2}$ equations in the following manner; namely, those in each equation being subject to σ equations of the form

$$U_1^2 + U_2^2 + \ldots + U_\sigma^2 = 1 \ldots\ldots\ldots\ldots\ldots\ldots (3).$$

and those in every pair (throughout both classes promiscuously) to $\frac{\sigma.\sigma-1}{2}$ equations of the form

$$U_1U_1' + U_2U_2' + \ldots + U_\sigma U_\sigma' = 0 \ldots\ldots\ldots\ldots (4),$$

we should then be ready to perform a transformation of which the two geometrical examples are particular cases. But, in general, we should have τ differential equations of the same form as (1), but with coefficients not satisfying equations (3) and (4); and for this system we should have to substitute such a system as (1) with coefficients that would satisfy them. It is only necessary to show that this is possible. That the system to be formed should be equivalent to the system given, the variables need not have the same values absolutely, but only so far as they are conditioned by the equations. Accordingly, divide the σ differentials into two sets of τ and ρ, and by proper eliminations find the values of each of the τ in terms of the ρ from each system of equations. Each pair of values must be equivalent, and as the ρ variables are independent and the equations of the first degree, the corresponding coefficients must be equal in each pair. This involves ρ conditions for each of the τ, and $\rho\tau$ for the whole τ. Besides these conditions the U's

must satisfy $\frac{\tau.\tau+1}{2}$ others of the forms (3) and (4), making $\tau\left(\rho+\frac{\tau+1}{2}\right)$ in all. But we have $\tau\sigma$ of these quantities to dispose of, and have therefore $\frac{\tau.\tau-1}{2}$ to spare.* In like manner, the $\rho\sigma$ V's are more than enough to satisfy the $\rho\tau$ conditions involving both U's and V's and the $\frac{\tau.\tau+1}{2}$ involving V's only.

If we start then from the four systems of equations (1, 2, 3, 4), and denote by D_σ and D_τ the determinants

$$\begin{vmatrix} U_{1.1} \; U_{1.2} \ldots\ldots U_{1.\sigma} \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ U_{\tau.1} \; U_{\tau.2} \ldots\ldots U_{\tau.\sigma} \\ V_{1.1} \; V_{1.2} \ldots\ldots V_{1.\sigma} \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ V_{\rho.1} \; V_{\rho.2} \ldots\ldots V_{\rho.\sigma} \end{vmatrix} \qquad \begin{vmatrix} U_{1.\rho+1} \; U_{1.\rho+2} \ldots U_{1.\sigma} \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ U_{\tau.\rho+1} \; U_{\tau.\rho+2} \ldots U_{\tau.\rho} \end{vmatrix},$$

it follows as a corollary from a fundamental theorem in the transformation of multiple integrals (see for instance the 18th of Moigno's *Lessons in the Integral Calculus*) that

$$\int\ldots\iint dv_1 dv_2 \ldots dv_\rho = \int\ldots\iint \frac{D_\sigma}{D_\tau}\, dx_1 dx_2 \ldots dx_\rho \;\ldots.\; (5),$$

in which formula the ρ x's have been chosen arbitrarily and the constituents of D_τ determined by the choice.

We must now eliminate the determinants. From (3) and (4) it follows immediately that $D_\sigma = \pm 1$. D_τ may be expressed in terms of partial differential coefficients with respect to the ρ variables we have chosen to make independent. For this purpose make constant all but one of these variables successively. With respect to x_r this will give the τ equations

$$U_{1.r} + U_{1.\rho+1}\frac{dx_{\rho+1}}{dx_r} + U_{1.\rho+2}\frac{dx_{\rho+2}}{dx_r} + \ldots + U_{1.\sigma}\frac{dx_\sigma}{dx_r} = 0,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$U_{\tau.r} + U_{\tau.\rho+1}\frac{dx_{\rho+1}}{dx_r} + U_{\tau.\rho+2}\frac{dx_{\rho+2}}{dx_r} + \ldots + U_{\tau.\sigma}\frac{dx_\sigma}{dx_r} = 0.$$

* One, when $\tau = 2$. For instance, one parameter is left variable by the conditions that two planes intersect at right angles in a given straight line.

Eliminate all the differential coefficients but that of $x_{\rho+t}$. Minor determinants being written as differential coefficients, the resulting equation is

$$U_{1.r}\frac{dD_\tau}{dU_{1.\rho+t}}+U_{2.r}\frac{dD_\tau}{dU_{2.\rho+t}}+\ldots+U_{\tau.r}\frac{dD_\tau}{dU_{\tau.\rho+t}}=-D_\tau\frac{dx_{\rho+t}}{dx_r}\ldots(6),$$

the left side is the determinant derived from D_τ, by substituting a column r for the column $\rho+t$. I write then, in a notation which will explain itself,

$$\frac{r}{\rho+t}.D_\tau=-\frac{dx_{\rho+t}}{dx_r}D_\tau.$$

Differentiate successively with respect to $U_{1.\rho+t'}$, $U_{2.\rho+t'}\ldots U_{\tau.\rho+t'}$, and multiply the results respectively by $U_{1.r'}$, $U_{2.r'}\ldots U_{\tau.r'}$. The effect upon the left-hand determinant is to substitute in it a column r' for the column $\rho+t'$. The effect upon the right-hand product is by (6) to multiply it by $-\frac{dx_{\rho+t'}}{dx_{r'}}$. In the above notation then

$$\frac{r'}{\rho+t'}.\frac{r}{\rho+t}.D_\tau=\frac{dx_{\rho+t'}}{dx_{r'}}\frac{dx_{\rho+t}}{dx_r}D_\tau.$$

Now, since the determinant $\frac{r'}{\rho+t'}.\frac{r}{\rho+t}.D_\tau$ differs only in sign from the determinant $\frac{r}{\rho+t'}.\frac{r'}{\rho+t}.D_\tau$, this is true of the products $\frac{dx_{\rho+t'}}{dx_{r'}}\frac{dx_{\rho+t}}{dx_r}$ and $\frac{dx_{\rho+t'}}{dx_r}\frac{dx_{\rho+t}}{dx_{r'}}$; so that apart from sign the derived determinant got by two substitutions, has two expressions in terms of partial differential coefficients, varying by the interchange of the quasi-denominators. The same argument being applied to the derived determinant got by n substitutions, it follows that *the product of the partial differential coefficients of n out of the τ variables, taken with respect to n out of the ρ variables, varies only in sign under all permutations of the quasi-denominators*, being otherwise equal to such derived determinant. Accordingly we may write, under all such permutations,

$$\frac{r^{(n)}}{\rho+t^{(n)}}\cdots\frac{r}{\rho+t}.D_\tau=(-)^n\frac{dx_{\rho+t^{(n)}}}{dx_{r^{(n)}}}\cdots\frac{dx_{\rho+t}}{dx_r}D_\tau\ldots(7).$$

Now all these derived determinants, together with D_τ make up the whole number of determinants in the expression

$$\left\| \begin{matrix} U_{1.1}, & U_{1.2} \ldots U_{1.\sigma} \\ U_{2.1}, & U_{2.2} \ldots U_{2.\sigma} \\ \multicolumn{2}{c}{\dotfill} \\ U_{\tau.1}, & U_{\tau.2} \ldots U_{\tau.\sigma} \end{matrix} \right\|.$$

But the sum of the squares of all these determinants is on the one hand by (3) and (4) equal to unity, and is on the other hand by what we have just seen, equal to

$$RD_\tau^2 \ldots\ldots\ldots\ldots\ldots\ldots (8),$$

where R is the sum of the squares of all terms expressed by the product on the right-hand of (7), only without repeating equivalent ones. The simplest way of expressing the last condition symmetrically, is to sum all terms and divide each by the number of equivalent ones in the group it belongs to. There are $1.2\ldots n$ equivalent permutations of each of the $\frac{\tau.\tau-1\ldots\tau-n+1}{1.2\ldots n}\;\frac{\rho.\rho-1\ldots\rho-n+1}{1.2\ldots n}$ not equivalent combinations expressed by the above product. If the characteristic Σ extend to all the combinations of any number of elements and to all their permutations, the value of R may be written

$$1+\Sigma\left(\frac{dx_{\rho+t}}{dx_r}\right)^2+\frac{1}{1.2}\Sigma\left(\frac{dx_{\rho+t}}{dx_r}\,\frac{dx_{\rho+t'}}{dx_{r'}}\right)^2+\ldots$$
$$+\frac{1}{1.2\ldots n}\Sigma\left(\frac{dx_{\rho+t}}{dx_r}\ldots\frac{dx_{\rho+t^{(n)}}}{dx_{r^{(n)}}}\right)^2+\ldots\ldots\ldots (9).$$

If then we make in (5) the substitution given by the relation (8), we get

$$\int\ldots\iint dv_1dv_2\ldots dv_\rho=\int\ldots\iint R^{\frac{1}{2}}\,dx_1dx_2\ldots dx_\rho \ldots\ldots\ldots (10),$$

n cannot be greater than the less of the two numbers τ and ρ. If τ is zero, the formula (9) is reduced to its first term: if ρ were zero, the whole question would be unmeaning. If either τ or ρ is unity, the products disappear from (9), and that must always be so when $\sigma=3$, as in the geometry of three rectangular co-ordinates. This simplifies the formula and disguises its character. But the same simplicity may be had in the general expression at the expense of explicitness. For just as the ordinary expression for the differential of surface may be symmetrically written

$$(dy^2dz^2+dz^2dx^2+dx^2dy^2)^{\frac{1}{2}},$$

provided it is borne in mind that the differential of the dependent variable means one thing in one term and another in the other; just so, observing that the differential coefficients being treated as real fractions, the equivalence spoken of at (7) will subsist identically, and the numerical coefficients will have no place, we may, subject to a like ambiguity of interpretation, say that R is the sum of the squares of the $\frac{\sigma.\sigma-1...\tau+1}{1.2...\rho}$ products of every ρ out of the σ differentials.

It has been made a fundamental principle above, that if the variables are unconditioned, their range is measured by the integral of the simple product of their differentials. That this should not be disturbed by passing from one system to another, we secured by the conditions (3) and (4), extensions of the conditions of rectangularity. But this was only so, because we started from a system of rectangular coordinates, and it is evident that what we have to secure in general is, that the two systems should be similar to one another. Any generalization then, like that here treated of, depends upon equations of the nature of equations of transformation of co-ordinates, together with conditions of the nature of conditions of similarity of co-ordinates.

Hadley, Middlesex,
Oct. 24, 1866.

ON THE SPHERICAL ELLIPSE, REFERRED TO TRILINEAR COORDINATES.

By Henry M. Jeffery, M.A.

1. The various forms of the equations to the plane conic, both trilinear and bilinear, have their counterparts in Spherical Geometry. In the one case α, β, γ, denote the sines of the perpendiculars drawn on the sides of the triangle of reference; in the other case, the Cartesian coordinates of a point are determined by the tangents of intercepts on two intersecting great circles. In the following investigation the sine will be frequently omitted before its arc, for brevity's sake.

2. The equation to the inscribed ellipse will be first discussed

$$\sqrt{(L\alpha)} + \sqrt{(M\beta)} + \sqrt{(N\gamma)} = 0.$$

3. To determine the foci of the curve.

Let f, g, h denote the sines of the coordinates, for the sake of distinctness, of the internal centre; f', g', h'; f'', g'', h'' of the two external centres, each centre being distant 90° from the others; $\rho_1 \rho_2$ the principal semi-diameters of the ellipse; ε the distance of either focus from the centre; α_1, α_2 the sines of the distances of the foci from BC. Then it will be found that

$$\alpha_1 + \alpha_2 = 2f \cos \varepsilon, \quad \alpha_1 \alpha_2 = \tan^2 \rho_2 \cos^2 \rho_1.$$

Hence $\alpha = f \cos \varepsilon \pm f' \sin \varepsilon$, where $f' = \dfrac{\sqrt{(f^2 - \sin^2 \rho_2)}}{\tan \varepsilon}$.

Similarly $\beta = g \cos \varepsilon \pm g' \sin \varepsilon$, $\gamma = h \cos \varepsilon \pm h' \sin \varepsilon$.

So also the external centre (f'', g'', h''), which is the pole of the major-diameter, is determined by equations of the form

$$f'' = \frac{\sqrt{(\sin^2 \rho_1 - f^2)}}{\sin \varepsilon}.$$

4. Let the coordinates of the centre make the angles θ, ϕ, χ with the principal diameter; let x_1, y_1; x_2, y_2; x_3, y_3 denote the tangents of the bilinear coordinates of the points of contact with the sides of ABC; it may be seen that

$$\cos \theta = \frac{f'}{\cos f} = \frac{x_1 \tan f}{\tan^2 \rho_1}, \quad \sin \theta = \frac{f''}{\cos f} = \frac{y_1 \tan f}{\tan^2 \rho_2},$$

with corresponding values for $\cos \phi$, $\sin \phi$, $\cos \chi$, $\sin \chi$. Hence

$$f^2 + f'^2 + f''^2 = 1.$$

5. Let $2n$ represent six times the volume of the pyramid formed by the radii of the sphere with the chords of the triangle ABC as edges, $2n = \sin b \sin c \sin A$. Also $2N = \sin B \sin C \sin a$, with a similar interpretation for the polar triangle.

The fundamental relation between the trilinear coordinates is

$$4n^2 = \Sigma\, (\alpha^2 \sin^2 a + 2\beta\gamma \sin b \sin c \cos a).$$

If δ denote the distance between two points,

$$4n^2 \cos \delta = \Sigma\, \{\alpha\alpha' \sin^2 a + (\beta\gamma' + \beta'\gamma) \sin b \sin c \cos a\}.$$

Hence, if O be the point (f, g, h),

$$2n \cos OA = f \sin a + g \sin b \cos c + h \sin c \cos b.$$

Now if F, G, H denote the coordinates of the point O taken with respect to $A'B'C'$, the polar triangle of ABC,

$$F = \cos OA, \quad G = \cos OB, \quad H = \cos OC.$$

Hence the preceding equations may be written conveniently

$$2n = afF + bgG + chH,$$

$$2n \cos\delta = af'F + bg'G + ch'H = afF' + bgG' + chH',$$

where a, b, c are written instead of $\sin a$, $\sin b$, $\sin c$; and f, g, h; f', g', h'; F, G, H; F', G', H', represent any the same two points.

6. To find the magnitude of the principal diameters of an ellipse inscribed in the triangle of reference.

Since the centres are distant 90° from each other, by § 5,

$$aFf' + bGg' + cHh' = 0,$$

$$aFf'' + bGg'' + cHh'' = 0.$$

These equations point to the same quadratic equation

$$aF\sqrt{(f^2 - \sin^2\rho)} + bG\sqrt{(g^2 - \sin^2\rho)} + cH\sqrt{(h^2 - \sin^2\rho)} = 0,$$

where $\qquad 2nF = af + bg\cos c + ch\cos b.$

Since $\;2n = aFf + bGg + cHh$, if $2\Sigma = aF + bG + cH$,

the quadratic may be presented in the form

$$\begin{aligned} 8\sin^4\rho\,\Sigma\,(\Sigma - aF)(\Sigma - bG)(\Sigma - cH) \\ - \sin^2\rho\,\{(a^2F^2 + b^2G^2 + c^2H^2)(a^2f^2F^2 + b^2g^2G^2 + c^2h^2H^2) \\ - 2(a^4F^4f^2 + b^4G^4g^2 + c^4H^4h^2)\} \\ + 8n(n - afF)(n - bgG)(n - chH) = 0. \end{aligned}$$

If the value of $\sin^2\rho_1$, $\sin^2\rho_2$ be positive, the centre is internal; if negative, the centre is external.

7. It may be simply proved that, if OM, ON be drawn perpendicularly on the sides of a triangle from a point O,

$$\sin A\cos OA = \cos OM\cos ON\sin MON = \cos g\cosh h\sin(\phi - \chi),$$

to use the preceding notation of § 4.

Hence the following relations connect the coordinates of the centres

$$g''h' - g'h'' = \sin A\cos OA = F\sin A,$$

$$\Sigma f(g''h' - g'h'') = \Sigma fF\sin A = 2N.$$

Similarly $\; gh'' - g''h = F'\sin A, \quad g'h - gh' = F''\sin A,$

$$GH'' - G''H = f'\sin a, \quad G'H - GH' = f''\sin a.$$

8. To determine the vertices of the ellipse.

If $p_1 p_2$, $p_3 p_4$ denote the sines of the perpendiculars from the extremities of the major and minor diameters on BC; it may be proved by bilinear coordinates that

$$p_1 + p_2 = 2f\cos\rho_1,\quad p_3 + p_4 = 2f\cos\rho_2,$$

$$p_1 p_2 \sec^2\rho_1 + p_3 p_4 \sec^2\rho_2 = f^2,$$

$$p_1 p_2 \operatorname{cosec}^2\rho_1 + p_3 p_4 \operatorname{cosec}^2\rho_2 = f^2(1 + \cot^2\rho_1 + \cot^2\rho_2) - 1.$$

Hence

$$p_1 p_2 = \frac{\sin^2\rho_1 - f^2}{\tan^2\rho_1 - \tan^2\rho_2}\tan^2\rho_2,\quad p_3 p_4 = \frac{f^2 - \sin^2\rho_2}{\tan^2\rho_1 - \tan^2\rho_2}\tan^2\rho_1.$$

The coordinates of the vertices are found from equations of the form

$$\alpha_1 = f\cos\rho_1 \pm f'\sin\rho_1,\quad \alpha_2 = f\cos\rho_2 \pm f''\sin\rho_2.$$

These equations may be written at once, as in § 3, from a diagram.

9. To determine the equations to the principal diameters.

The major diameter may be represented by the various forms of the same equation

$$\Sigma\alpha(hg' - h'g) = 0,\quad \Sigma\frac{\alpha}{f}(x_2 - x_3) = 0,\ \text{ or }\ \Sigma a\alpha F'' = 0;$$

the minor diameter by the corresponding forms

$$\Sigma\alpha(hg'' - h''g) = 0,\quad \Sigma\frac{\alpha}{f}(y_2 - y_3) = 0,\ \text{ or }\ \Sigma a\alpha F' = 0.$$

10. To find the intercepts between a centre and the feet of the perpendiculars drawn upon the principal diameters from a given point.

Let P be the given point, O the centre (f, g, h), PM, PN the perpendiculars on the principal diameters OM, ON; then

$$\tan OM = \cos POM \tan OP;\quad \tan ON = \sin POM \tan OP.$$

Let x, y denote these intercepts OM, ON; α_1, β_1, γ_1 the sines of the trilinear coordinates of P. Then the equations to OP and the major diameter are

$$\Sigma\alpha(\gamma_1 g - \beta_1 h) = 0,\quad \Sigma\alpha(h'g - g'h) = 0.$$

It may be proved that

$$\sin POM = \frac{\Sigma\alpha_1(h'g - g'h)}{2N\sin OP},$$

where $$2N = a\sin B\sin C.$$

Hence $\qquad 2N\cos OP\tan y = \Sigma\alpha_1(g'h - gh')$ .... (1),

where $\quad 2n\cos OP = a\alpha_1 F + b\beta_1 G + c\gamma_1 H$, (see § 5).

Similarly $\quad 2N\cos OP\tan x = \Sigma\alpha_1(h''g - hg'')$ .... (2).

Also, since $g'h - gh' = F''\sin A$, these equations (1) and (2) may be written

$$2n\cos OP\tan y = \Sigma a\alpha_1 F'' = 2n\cos O_2 P,$$

$$2n\cos OP\tan x = \Sigma a\alpha_1 F' = 2n\cos O_1 P,$$

if O_1, O_2 be the two external centres. These results may be independently obtained by using bilinear coordinates.

11. In the general equation to a ternary quadric

$$u\alpha^2 + v\beta^2 + w\gamma^2 + 2u'\beta\gamma + 2v'\gamma\alpha + 2w'\alpha\beta = 0,$$

to investigate the coordinates of the three centres, internal and external.

They are found from the consideration that the polars of these points with respect to the conic are also *quadrantal* polars, *i.e.* are situated at the distance of 90° from their poles.

If a centre be defined by (f, g, h) as its coordinates in the primitive triangle ABC, and by (F, G, H) in the polar triangle $A'B'C'$, its polar with respect to the conic $\phi(\alpha, \beta, \gamma)$ is defined by the equation

$$\frac{d\phi}{df}\alpha + \frac{d\phi}{dg}\beta + \frac{d\phi}{dh}\gamma = 0.$$

The quadrantal pole of this line has for its equations

$$\frac{f}{\frac{d\phi}{df} - \frac{d\phi}{dg}\cos C - \frac{d\phi}{dh}\cos B} = \frac{g}{-\frac{d\phi}{df}\cos C + \frac{d\phi}{dg} - \frac{d\phi}{dh}\cos A}$$

$$= \frac{h}{-\frac{d\phi}{df}\cos B - \frac{d\phi}{dg}\cos A + \frac{d\phi}{dh}}.$$

By expressing these conditions in the form of a determinant, we may obtain relations analogous to those in Plane Geometry,

$$\lambda f = af(Ua + W'b\cos c + V'c\cos b)$$

$$+ bg(Ua\cos c + W'b + V'c\cos a) + ch(Ua\cos b + W'b\cos a + V'c)$$

with similar expressions for λg, λh, (λ being an indeterminate multiplier). These conditions point to the following geometrical relations.

If O be a centre, $T_1T_2T_3$ the poles of the sides of ABC with respect to the conic, it will be recognized that, by the last equation and by § 5,

$$\frac{f}{Q_1 \cos OT_1} = \frac{g}{Q_2 \cos OT_2} = \frac{h}{Q_3 \cos OT_3};$$

since U, W', V' are of the coordinates of T_1, and

$$Q_1^2 = U^2a^2 + W'^2b^2 + V'^2c^2$$
$$+ 2W'V'bc \cos a + 2UV'ac \cos b + 2UW'ab \cos c.$$

Each of these ratios is equal to

$$\frac{uf + w'g + v'h}{KaF} = \frac{\phi(f, g, h)}{2nK}$$

$$= \frac{f}{aUF + bW'G + cV'H} = \frac{2n}{\Sigma(a^2UF^2 + 2bcU'GH)},$$

where K represents the Hessian $H(u)$. The geometrical significance of these ratios I will proceed to explain.

12. Since the equation to the polar of the centre (f, g, h) is

$$\alpha \frac{d\phi}{df} + \beta \frac{d\phi}{dg} + \gamma \frac{d\phi}{dh} = 0,$$

let p, q, r be the sines of the perpendiculars drawn from the angular points of ABC on this polar; in fact, let p, q, r be its tangential coordinates,

$$\frac{ap}{\frac{d\phi}{df}} = \frac{bq}{\frac{d\phi}{dg}} = \frac{cr}{\frac{d\phi}{dh}}.$$

But $\quad p = \cos OA = F$, $q = \cos OB = G$, $r = \cos OC = H$.

Hence, if each of the ratios in § 11 be denoted by $\dfrac{2n}{\mu K}$,

$$\mu(uf + w'g + v'h) = 2naF = a(af + bg \cos c + ch \cos b) \text{ by § 5.}$$

This theorem might be adopted at once to determine the centres.

13. To determine the coordinates of the centres.

The eliminant of the three equations of condition leads to a cubic in μ.

$$(\mu u - a^2)f + (\mu w' - ab \cos c)g + (\mu v' - ac \cos b)h = 0 \ldots (1).$$
$$(\mu w' - ab \cos c)f + (\mu v - b^2)g + (\mu u' - bc \cos a)h = 0 \ldots (2).$$
$$(\mu v' - ac \cos b)f + (\mu u' - bc \cos a)g + (\mu w - c^2)h = 0 \ldots (3).$$

ELEMENTARY MATHEMATICAL WORKS

BY THE

RIGHT REV. J. W. COLENSO, D.D.

LORD BISHOP OF NATAL.

ARITHMETIC designed for the use of SCHOOLS; to which is added a Chapter on DECIMAL COINAGE. New Edition, thoroughly revised, with the addition of Notes and Examination Papers. 12mo. 4s. 6d.

KEY to COLENSO'S ARITHMETIC for SCHOOLS; containing full SOLUTIONS of all the Exercises: adapted to the above Revised and Enlarged Edition of 1864. By the Rev. J. HUNTER, M.A. 12mo. 5s.

COLENSO'S ELEMENTARY ARITHMETIC, for the use of NATIONAL, ADULT, and COMMERCIAL SCHOOLS; consisting of a Text-Book of Arithmetic, accompanied by a Series of easy and amusing Examples arranged progressively in order of difficulty. 18mo. 1s. 9d.; or with ANSWERS, 2s. 3d.

COLENSO'S ELEMENTARY ARITHMETIC for National Schools is divided into FIVE PARTS, which are sold separately as follows:—

I. TEXT BOOK, price 6d.
II. EXAMPLES, PART I. *Simple Arithmetic*, price 4d.
III. EXAMPLES, PART II. *Compound Arithmetic*, price 4d.
IV. EXAMPLES, PART III. *Fractions, Decimals, Duodecimals*, &c. price 4d.
V. ANSWERS to the EXAMPLES, with SOLUTIONS of the more difficult QUESTIONS, price 1s.

COLENSO'S ARITHMETICAL TABLES, on a Card, Price 1d.

THE ELEMENTS of ALGEBRA, designed for the use of SCHOOLS, PART I. containing the Simpler Parts of the Science, suitable for general School purposes, and as required for the attainment of an ordinary B.A. degree. 12mo. 4s. 6d. KEY, 5s.

HUNTER'S EXAMINATION-QUESTIONS ON COLENSO'S ALGEBRA, PART I. with Graduated Exercises, Problems, &c. 12mo. 2s. 6d.

COLENSO'S ELEMENTS OF ALGEBRA, designed for the use of SCHOOLS, PART II. adapted to suit the wants of more advanced Students; containing the higher parts of the subject, with an APPENDIX of more difficult Miscellaneous Examples and Equation Papers. 12mo. 6s. KEY, 5s.

COLENSO'S MISCELLANEOUS EXAMPLES AND EQUATION PAPERS, from PARTS I. and II. of the *Algebra*; with the ANSWERS. 12mo. 2s. 6d.

COLENSO'S ELEMENTS OF ALGEBRA, adapted for the use of NATIONAL and ADULT SCHOOLS; containing numerous easy Examples and Questions under every Rule, with a Selection of Miscellaneous Exercises for Practice. 18mo. 1s. 6d. KEY, 2s. 6d.

COLENSO'S ELEMENTS OF ALGEBRA, adapted for the use of TEACHERS and STUDENTS in the UNIVERSITY; being a Large-Paper Edition of the above. 8vo. 12s. 6d. KEY, 12mo. 7s. 6d.

PLANE TRIGONOMETRY, PART I. comprising the Measurement of Lines and Angles, the Numerical Values of the Trigonometrical Ratios; with the use of Logarithms and the Exponential Theorem. 12mo. 3s. 6d. KEY, 3s. 6d.

COLENSO'S PLANE TRIGONOMETRY, PART II. comprising the Summation of Series, the Trigonometrical Solution of Equations, and Miscellaneous Problems. 12mo. 2s. 6d. KEY, 5s.

⁎ The KEYS to the TWO PARTS of the TRIGONOMETRY, together, price 8s. 6d.

THE ELEMENTS of EUCLID (as studied in the Universities); with a large Collection of GEOMETRICAL PROBLEMS. 18mo. 4s. 6d.; or with a KEY to the PROBLEMS, 6s. 6d.

COLENSO'S GEOMETRICAL PROBLEMS and KEY. 18mo. 3s. 6d.

COLENSO'S GEOMETRICAL PROBLEMS separately, for the use of Schools where other Editions of *Euclid* may be employed. 18mo. 1s.

London: LONGMANS, GREEN, and Co., Paternoster Row.

No. 32. May, 1867.

THE

QUARTERLY JOURNAL

OF

PURE AND APPLIED

MATHEMATICS.

EDITED BY

J. J. SYLVESTER, M.A., F.R.S.,

PROFESSOR OF MATHEMATICS IN THE ROYAL MILITARY ACADEMY, WOOLWICH; AND

N. M. FERRERS, M.A.,

FELLOW AND TUTOR OF GONVILLE AND CAIUS COLLEGE, CAMBRIDGE:

ASSISTED BY

G. G. STOKES, M.A., F.R.S.,

LUCASIAN PROFESSOR OF MATHEMATICS IN THE UNIVERSITY OF CAMBRIDGE;

A. CAYLEY, M.A., F.R.S.,

SADLERIAN PROFESSOR OF MATHEMATICS IN THE UNIVERSITY OF CAMBRIDGE; AND

M. HERMITE,

CORRESPONDING EDITOR IN PARIS.

ὅ τι οὐσία πρὸς γένεσιν, ἐπιστημὴ πρὸς πίστιν καὶ διάνοια πρὸς εἰκασίαν ἐστι.

LONDON:

LONGMANS, GREEN, AND CO.,

PATERNOSTER ROW.

1867.

W. METCALFE, PRINTER, *PRICE FIVE SHILLINGS.* GREEN STREET, CAMBRIDGE.

CONTENTS.

NOTICES TO CORRESPONDENTS.

The following Papers have been received:

Mr. WALTON, "On the Symbol of Operation $x\frac{d}{dx}$;" "On the Debility of Large Trees and Animals;" "On Biangular Coordinates;" "A Demonstration of a Proposition in Euclid's Elements;" and "On an Equation of Finite Differences."

PROFESSOR CAYLEY, "Specimen Table $M\equiv a^\alpha b^\beta$ (mod. N) for any Prime or Composite Modulus;" "Tables of the Binary Cubic Forms for the Negative Determinants, $\equiv 0$ (Mod. 4), from -4 to -400; and $\equiv 1$ (Mod. 4), from -3 to -99; and for five Irregular Negative Determinants;" and "On a certain Envelope depending on a Triangle Inscribed in a Circle."

Mr. J. BLISSARD, "On the Properties of the $\Delta^m 0^n$ Class of Numbers."

Mr. H. M. JEFFERY, "On Conics, Plane and Spherical, referred to Three-Point Tangential Coordinates."

Mr. S. ROBERTS, "On the Centres of Curves and Surfaces;" and "On the Centres of Mean Distances of Certain Points of Intersection of Curves and Surfaces."

Mr. J. HORNER, "On Triads of Once-Paired Elements;" and "On certain Criteria of Imaginary and Equal Roots."

Mr. C. NIVEN, "On some Theorems connected with the Wave-Surfaces."

PROFESSOR A. H. CURTIS, "On the Equilibrium of a Heavy Body bounded by a Surface of Revolution, and resting on a Rough Surface also of Revolution."

Mr. J. C. TURNBULL, "The Loci of Centres."

Notice to Subscribers.—The first Eight Volumes of the 'Journal,' bound in calf, gilt, or any of the back Numbers, can be had of the Publishers, or at the Printing Office, Green Street, Cambridge.

No. 33 will be published in July, 1867.

The cubic is identical with the well-known discriminating equation, when $u' = v' = w' = 0$. Its form is

$$(\mu u - a^2)(\mu v - b^2)(\mu w - c^2) - (\mu u - a^2)(\mu u' - bc\cos a)^2$$
$$- (\mu v - b^2)(\mu v' - ac\cos b)^2 - (\mu w - c^2)(\mu w' - ab\cos c)^2$$
$$+ 2(\mu u' - bc\cos a)(\mu v' - ac\cos b)(\mu w' - ab\cos c) = 0.$$

It may be reduced to the form

$$K\mu^3 - \Sigma(Ua^2 + 2U'bc\cos a)\mu^2$$
$$+ \Sigma(u - 2u'\cos A)a^2b^2c^2\mu - 4n^2a^2b^2c^2 = 0.$$

By various eliminations effected in the three equations, we find that

$$\frac{f^2}{(\mu v - b^2)(\mu w - c^2) - (\mu u' - bc\cos a)^2}$$
$$= \frac{g^2}{(\mu w - c^2)(\mu u - a^2) - (\mu v' - ac\cos b)^2}$$
$$= \frac{h^2}{(\mu u - a^2)(\mu v - b^2) - (\mu w' - ab\cos c)^2}$$
$$= \frac{gh}{(\mu v' - ac\cos b)(\mu w' - ab\cos c) - (\mu u - a^2)(\mu u' - bc\cos a)}$$
$$= \frac{\phi(f, g, h)}{K\left(\dfrac{dM}{d\mu}\right)} = \frac{\phi(f, g, h)}{K(\mu - \mu_1)(\mu - \mu_2)},$$

if $KM = 0$ denote the cubic.

For the several values of $\mu\mu_1\mu_2$ determined by the preceding cubic, we have corresponding values of the coordinates of the centre. The determination of their actual values requires the aid of the fundamental equation

$$\Sigma(a^2f^2 + 2bcgh\cos a) = 4n^2.$$

The discrimination of the centres will be explained subsequently.*

* Since the equation to a spherical ellipse $\phi(\alpha, \beta, \gamma) = 0$ is identical in form with that to the corresponding cone $\phi\left(\dfrac{x}{\sin a}, \dfrac{y}{\sin b}, \dfrac{z}{\sin c}\right) = 0$, when referred to oblique axes, the coordinates of the centres of the ellipse are the same as the equations to the principal diameters of the cone.—Gregory's *Solid Geometry*, p. 89. If a cone be referred to three conjugate diameters as oblique axes, the triangle of reference, which is the trace on the sphere of the conjugate diametral planes, is self-conjugate with respect to the spherical conic, which is the trace of the cone.

14. By the geometry of the sphere, it appears that

$$\frac{f}{\cos OT_1} : \frac{f'}{\cos O'T_1} : \frac{f''}{\cos O''T_1} :: 1 : \frac{1}{\tan^2\rho_1} : \frac{1}{\tan^2\rho_2},$$

if O, O', O'' denote the three centres.

Hence, by § 11,

$$\mu : \mu_1 : \mu_2 :: 1 : \tan^2\rho_1 : \tan^2\rho_2.$$

This theorem enables us to ascertain the relative magnitude of the tangents of the principal diameters, whichever centre be first selected.

This conclusion may be illustrated by reference to bilinear coordinates, whereby the same ellipse is represented by the three equations referred to its several centres

$$\frac{x^2}{\tan^2\rho_1} + \frac{y^2}{\tan^2\rho_2} = 1;$$

$$x^2\tan^2\rho_1 - y^2\frac{\tan^2\rho_1}{\tan^2\rho_2} = 1; \quad y^2\tan^2\rho_2 - x^2\frac{\tan^2\rho_2}{\tan^2\rho_1} = 1.$$

15. To determine the trilinear equation of the supplementary conic, or the reciprocal polar of the primitive.

The equation to the tangent at a point (x, y, z) of the primitive is

$$\alpha\frac{d\phi}{dx} + \beta\frac{d\phi}{dy} + \gamma\frac{d\phi}{dz} = 0.$$

Now, by § 5, if (X, Y, Z) be the corresponding point in the supplementary conic, whose coordinates are estimated by the polar triangle $A'B'C'$, its quadrantal polar is defined by the equation

$$a\alpha X + b\beta Y + c\gamma Z = 0.$$

Equate coefficients, use an indeterminate multiplier, and multiply by U, W', V' in succession,

$$UaX + W'bY + V'cZ \propto xH(u).$$

Substitute this value of x and corresponding values of y, z in the primitive quadric, and the resulting equation is the ternary quadric

$$(U, V, W, U', V', W' \between aX, bY, cZ)^2 = 0.$$

It is called a supplementary conic to the primitive, because their principal diameters are reciprocally supplementary.

These conics are concentric: so that the ratio, obtained in §11,

$$\frac{f}{aUF+bW'G+cV'H}=\frac{2n}{\mu K}$$

denotes that the tangential coordinates of a quadrantal polar of a centre of either conic, taken with reference to the polar triangle, are proportional to the primitive trilinear coordinates of that centre. See §12.

16. The discriminating cubic for the reciprocal polar is

$$K^2\mu'^3-K\mu'^2\Sigma(u+2u'\cos a')+\mu'\Sigma(Ua'^2-2U'b'c'\cos A')-4n'^2=0,$$

estimated with reference to the polar triangle.

μ' is taken similarly to μ, and is connected by the relation

$$\mu\mu'=\frac{4n^2}{K}.$$

17. If the ellipse degenerate into a small circle, the coordinates of the centre are

$$\frac{f}{u-w'\cos C-v'\cos B-\frac{4n^2}{\mu}}$$

$$=\frac{g}{w'-u\cos C-v'\cos A}$$

$$=\frac{h}{v'-u\cos B-w'\cos A}.$$

In this case, the external centres are indefinite.

Hence the values of f^2, ..., gh, ... obtained in §13 are indefinite. The denominators vanish,

$$\mu^2V-\mu(uc^2+wa^2-2v'ac\cos b)+a^2b^2c^2=0,$$

$$\mu^2W-\mu(va^2+ub^2-2w'ab\cos c)+a^2b^2c^2=0,$$

$$\mu^2U'-\mu(w'ac\cos b+v'ab\cos c-u'a^2-ubc\cos a)-a^2b^2c^2\cos A=0.$$

Hence we may obtain three conditions of the form

$$\frac{\mu}{a^2}(V\cos A+W\cos A+2U')$$

$$=-u\frac{\cos A\cos^2B+\cos A\cos^2C+2\cos B\cos C}{\sin^2A}+v\cos A+w\cos A$$

$$-2u'+2v'\cos C+2w'\cos B.$$

The equation to the small circle may be written in the form

$$\alpha(\mu u - a^2) + \beta(\mu w' - ab\cos c) + \gamma(\mu v' - ac\cos b) = 2n\sqrt{(a^2 - \mu u)}.$$

This result may be obtained also directly by assuming that the equation to a small circle may be written under the form $\lambda\alpha + \mu\beta + \nu\gamma = d$, and determining the relations between the coefficients that $\phi(\alpha, \beta, \gamma) - 4n^2d^2 = 0$ may be a complete square. The coordinates of the centre of the circle are found by considering it the quadrantal pole of the great circle, to which it is parallel, viz.

$$\alpha(\mu u - a^2) + \beta(\mu w' - ab\cos c) + \gamma(\mu v' - ac\cos b) = 0.$$

Ex. To investigate the equation to a circle, with respect to which the triangle of reference is self-conjugate,—its centre and radius.

The equations of condition are of the form

$$\mu u = \frac{a^2bc\cos A}{\cos a}.$$

The required equation is

$$\alpha^2\frac{\sin 2A}{\cos a} + \beta^2\frac{\sin 2B}{\cos b} + \gamma^2\frac{\sin 2C}{\cos c} = 0.$$

It may be also written in the form

$$\alpha\tan a + \beta\tan b + \gamma\tan c = 2n\sqrt{(\sec a\sec b\sec c)}.$$

The coordinates of the centre are thus related

$$\alpha\cos A = \beta\cos B = \gamma\cos C.$$

As in Plane Geometry, the centre is the intersection of perpendiculars drawn from the angular points of ABC to the opposite sides. If ρ be the angular radius,

$$\cos\rho = \frac{2n\sqrt{(\sec\alpha\sec\beta\sec\gamma)}}{\sqrt{\{\Sigma(\tan^2 a - 2\tan b\tan c\cos A)}}$$

$$= \frac{2n}{\sqrt{\{abc(\tan a\cos B\cos C + \tan b\cos A\cos C + \tan c\cos A\cos B)\}}}.$$

18. To determine the foci.

If p_1p_2Pp denote the sines of the respective perpendiculars on any line drawn from the foci of a conic, its own pole (T_1) and a centre (O), $\rho_1\rho_2$ the principal semi-diameters corresponding to that centre, it may be shewn by bilinear coordinates, that

$$p_1p_2\sec^2\rho_1 = \tan^2\rho_2 + pP\sec OT_1.$$

Let BC be the fixed line, its pole is known from the conditions

$$\frac{\alpha}{U}=\frac{\beta}{W'}=\frac{\gamma}{V'}=\frac{2n}{Q_1}.$$

Hence, if the centre (f, g, h) lie on the major diameter, one of the equations to determine the foci is (by §11),

$$\alpha^2-2\alpha f\cos\varepsilon+\frac{4n^2U}{\mu K}\cos^2\rho_1+\tan^2\rho_2\cos^2\rho_1=0.$$

As in §3, since $\alpha=f\cos\varepsilon\pm f'\sin\varepsilon$, where (f', g', h') is the other centre lying on the major diameter, it follows that

$$f'^2\tan^2\varepsilon=f^2-\sin^2\rho_2-\frac{4n^2U}{\mu K}\cos^2\rho_2,$$

so also $$f''^2\sin^2\varepsilon=\sin^2\rho_1-f^2+\frac{4n^2U}{\mu K}\cos^2\rho_1,$$

since $$f^2+f'^2+f''^2=1.$$

19. To find the magnitude of the principal diameters.

By proceeding as in §6, we obtain the necessary quadratic equation

$$\Sigma aF\sqrt{\left(f^2-\sin^2\rho-\frac{4n^2U}{\mu K}\cos^2\rho\right)}=0.$$

In the case of a small circle,

$$\cos^2\rho=\frac{1-f^2}{1-\dfrac{4n^2U}{\mu K}}=\ldots=\ldots.$$

20. Hence we can discriminate the different centres.

According as the value of $\tan^2\rho_1\ \tan^2\rho_2$ in the preceding equation, corresponding to the particular values of (f, g, h) and consequently of μ in the discriminating cubic, is positive or negative, the selected centre (f, g, h) is internal or external. The simplest course to distinguish the external centres will be to ascertain first the internal centre by successive trials.

21. To find the equations to the principal diameters or quadrantal polars of the three centres.

The quadrantal polar of (f, g, h) is, by §5, defined to be $\Sigma a\alpha F=0$. So also the polars of the other centres are $\Sigma a\alpha F'=0$, $\Sigma a\alpha F''=0$. They may be also written

$$\Sigma\alpha\frac{d\phi}{df}=0,\ \Sigma\alpha\frac{d\phi}{df'}=0,\ \Sigma\alpha\frac{d\phi}{df''}=0,$$

from the condition of being polars of the centres, see also § 11. Implicit forms may be also found, as in Plane Geometry.

Let δ denote the distance of any point in the principal diameter (α, β, γ) from the centre (f, g, h), then if θ, ϕ, χ be defined, as in § 4,

$$\frac{\alpha - f\cos\delta}{\cos\theta\cos f} = \frac{\beta - g\cos\delta}{\cos\phi\cos g} = \frac{\gamma - h\cos\delta}{\cos\chi\cos h}.$$

Also, if $(\alpha', \beta', \gamma')$ be a focus, since $\alpha' = f\cos\varepsilon \pm f'\sin\varepsilon$,

$$\frac{f'}{\cos\theta\cos f} = \frac{g'}{\cos\phi\cos g} = \frac{h'}{\cos\chi\cos h}.$$

Hence, by substitution in the identity

$$\Sigma\cos^2\theta(\cos^2\phi - \cos^2\chi) = 0,$$

we shall obtain an equation independent of ρ, (except that the selected centre (f, g, h) is itself a function of ρ),

$$\Sigma(\alpha - f\cos\delta)^2(g'^2\cos^2 h - h'^2\cos^2 g) = 0,$$

where $$2n\cos\delta = \Sigma a\alpha F,$$

or $$\Sigma(\alpha - f\cos\delta)^2\left\{g^2 - h^2 - [(1-h^2)V - (1-g^2)W]\frac{4n^2}{\mu K}\right\} = 0.$$

22. To transform the equation to a spherical ellipse from bilinear to trilinear coordinates.

Simple relations connect these coordinate systems.

Let O, O', O'' be the three centres, internal and external, P any point, OM, ON its bilinear coordinates; α, β, γ its trilinear coordinates; then it may be recognized from a figure, that

$$\tan OM = \frac{\cos O'P}{\cos OP};\ \tan ON = \frac{\cos O''P}{\cos OP}.$$

And since, by § 5,

$$2n\cos OP = \Sigma(a\alpha F'),\ 2n\cos O'P = \Sigma(a\alpha F''),$$

we have after substitution in the bilinear equation to the curve

$$\frac{\cos^2 O'P}{\tan^2\rho_1} \pm \frac{\cos^2 O''P}{\tan^2\rho_2} = \cos^2 OP,$$

$$\frac{(\Sigma a\alpha F')^2}{\tan^2\rho_1} \pm \frac{(\Sigma a\alpha F'')^2}{\tan^2\rho_2} = (\Sigma a\alpha F)^2.$$

This transformation is important, whenever a curve is referred to a centre as the origin of bilinear coordinates.

23. To prove that this last form may be identified with the general quadric form $\phi(\alpha, \beta, \gamma) = 0$.

Examine the coefficient of α^2,

$$\frac{a^2F'^2}{\tan^2\rho_1} + \frac{a^2F''^2}{\tan^2\rho_2} - a^2F^2.$$

By § 12, this is equal to

$$\frac{\mu_1}{4n}\frac{aF'}{\tan^2\rho_1}\frac{d\phi}{df'} + \frac{\mu_2}{4n}\frac{aF''}{\tan^2\rho_2}\frac{d\phi}{df''} - \frac{\mu}{4n}aF\frac{d\phi}{df},$$

(by § 14) $$= -\frac{\mu}{4n}\left\{aF\frac{d\phi}{df} - aF'\frac{d\phi}{df'} - aF''\frac{d\phi}{df''}\right\},$$

(by § 7)

$$= -\frac{\mu}{4N}\left\{(g''h' - g'h'')\frac{d\phi}{df} + (gh'' - g''h)\frac{d\phi}{df'} + (g'h - gh')\frac{d\phi}{df''}\right\}$$

$$= -\frac{\mu}{2N}u\Sigma f(g''h' - g'h'') = -\mu u.$$

Each of the other coefficients may be similarly identified.

24. To determine the geometrical signification of each coefficient in the general quadric form.

By § 23, $$-\frac{\mu u}{a^2} = \frac{F'^2}{\tan^2\rho_1} + \frac{F''^2}{\tan^2\rho_2} - F^2.$$

And since

$$\frac{F'}{F} = \frac{\cos O'A}{\cos OA} = \tan x_1, \quad \frac{F''}{F} = \frac{\cos O''A}{\cos OA} = \tan y_1,$$

if $\tan x_1$, $\tan y_1$ be the bilinear coordinates of A,

$$-\frac{\mu u}{a^2F^2} = \frac{\tan^2 x_1}{\tan^2\rho_1} + \frac{\tan^2 y_1}{\tan^2\rho_2} - 1 = \frac{p_1}{FP},$$

if p_1, P denote the sines of the perpendiculars drawn from A and the centre O on the polar of A. Similarly

$$-\frac{\mu u'}{bcGH} = \frac{\tan x_2 \tan x_3}{\tan^2\rho_1} + \frac{\tan y_2 \tan y_3}{\tan^2\rho_2} - 1$$

$$= \frac{r_2}{HQ} = \frac{q_3}{GR},$$

if $p_2q_2r_2$ be the tangential coordinates of the polar of B, $p_3q_3r_3$ of the polar of C, and P, Q, R the sines of the perpendiculars from O on the polars of A, B, C.

25. The same conclusion may be derived independently. The polar of A is

$$u\alpha + w'\beta + v'\gamma = 0.$$

Its tangential coordinates p, q, r, have the following values:

$$\frac{ap_1}{u} = \frac{bq_1}{w'} = \frac{cr_1}{v'} = \frac{2n}{\sqrt{\{\Sigma(u^2 - 2w'v'\cos A)\}}} = \frac{\mu P}{aF},$$

$$\frac{ap_2}{w'} = \frac{bq_2}{v} = \frac{cr_2}{u'} = \frac{\mu Q}{bG},$$

$$\frac{ap_3}{v'} = \frac{bq_3}{u'} = \frac{cr_3}{w} = \frac{\mu R}{cH}.$$

26. Hence, the general quadric may be written to express the values of its coefficients

$$\Sigma \frac{aF}{P} \alpha (p_1 a\alpha + q_1 b\beta + r_1 c\gamma) = 0.$$

It may be also otherwise expressed, if p, q, r be the tangential coordinates of the quadrantal polar of the selected centre, and $\omega_1\omega_2\omega_3$ be the inclinations of the polars of A, B, C to that quadrantal polar

$$\Sigma ap\alpha \sec\omega_1 (p_1 a\alpha + q_1 b\beta + r_1 c\gamma) = 0.$$

27. These propositions enable us to express various geometrical conditions.

(1) If $u = 0$, A is a point on the curve.

Hence, if $\frac{u'}{\alpha} + \frac{v'}{\beta} + \frac{w'}{\gamma} = 0$, the conic is described about ABC.

(2) If $u' = 0$, the polars of B, C pass through C, B respectively.

Hence, if $u\alpha^2 + v\beta^2 + w\gamma^2 = 0$, ABC is self-conjugate to the conic.

(3) If $U = 0$, BC is tangential.

Hence, if $\sqrt{(u^{\frac{1}{2}}\alpha)} + \sqrt{(v^{\frac{1}{2}}\beta)} + \sqrt{(w^{\frac{1}{2}}\gamma)} = 0$, the conic is inscribed in ABC.

(4) If $U' = 0$, the pole of AC lies on AB, and that of AB on AC.

(5) The equation to the conic, which touches the two sides of ABC in the extremities of the base, is

$$u\alpha^2 + 2u'\beta\gamma = 0.$$

28. To find the equations to the asymptotic circles.

These circles touch the ellipse at the extremities of the principal diameters, and intersect in the corresponding external centres. They are four in number. If the curve be referred to its internal centre

$$\frac{(\Sigma\alpha a F')^2}{\tan^2\rho_1}+\frac{(\Sigma\alpha a F'')^2}{\tan^2\rho_2}=(\Sigma\alpha a F)^2,$$

the asymptotes are defined by the equations

$$\Sigma\alpha a F'=\pm\tan\rho_1\Sigma\alpha a F;\quad \Sigma\alpha a F''=\pm\tan\rho_2\Sigma\alpha a F.$$

Cor. The conjugate ellipses are expressed by the forms

$$\frac{(\Sigma\alpha a F')^2}{\tan^2\rho_1}-\frac{(\Sigma\alpha a F'')^2}{\tan^2\rho_2}=\pm(\Sigma\alpha a F)^2.$$

29. In studying the doctrine of Spherical Coordinates, the principle of duality or reciprocation must be ever kept in view. Even in the fundamental equation, the trilinear coordinates of a point, referred to both the primitive and polar triangles are blended conveniently. (See the forms for $2n$ and $2n\cos\delta$ in § 5). The ratios, which determine the coordinates of the centre of an ellipse, alternately employ both primitive and polar triangles, and even point to the reciprocal polar of the ellipse. (See § 11, 15). This connection will be rendered more evident by applying tangential coordinates to this branch of the subject, as I next propose to explain.

Cheltenham, *Oct.*, 1866.

NOTE ON THE LUNAR THEORY.

By William Walton, M.A., Trinity College.

In the computation of the approximate expression for $\dfrac{T}{h^2u^3}$ in terms of the moon's longitude, given in Godfray's *Treatise on the Lunar Theory*, u^{-4} is replaced by the expression

$$a^{-4}\{1-4e\cos(c\theta-\alpha)+10e^2\cos^2(c\theta-\alpha)\}:$$

now the value of u, free from the effects of disturbance, is given by the equation, to the same order of small quantities,

$$u = a\{1 + e\cos(c\theta - \alpha) - \tfrac{1}{4}k^2\cos 2(g\theta - \gamma)\},$$

and consequently, adopting the same general course of reasoning, we ought to have

$$u^{-4} = a^{-4}\{1 - 4e\cos(c\theta - \alpha) + 10e^2\cos^2(c\theta - \alpha) + k^2\cos 2(g\theta - \gamma)\}.$$

If this value of u^{-4} be made use of in our approximations, we shall have, retaining in the process important terms only,

$$\begin{aligned}\frac{T}{h^2u^3} &= -\tfrac{3}{2}m^2\{1 - 4e\cos(c\theta - \alpha) + 5e^2 + 5e^2\cos(2c\theta - 2\alpha) \\ &\qquad + k^2\cos 2(g\theta - \gamma)\}\sin\{(2 - 2m)\theta - 2\beta\} \\ &= -\tfrac{3}{2}m^2[\sin\{(2-2m)\theta - 2\beta\} - 2e\sin\{(2-2m-c)\theta - 2\beta + \alpha\} \\ &\qquad + \frac{5e^2}{2}\sin\{(2-2m-2c)\theta - 2\beta + \alpha\} \\ &\qquad + \tfrac{1}{2}k^2\sin\{(2-2m-2g)\theta - 2\beta + 2\gamma\}].\end{aligned}$$

Hence also

$$\begin{aligned}\int\frac{T}{h^2u^3}\,d\theta &= \tfrac{3}{4}m^2\cos\{(2-2m)\theta - 2\beta\} \\ &\quad - 3m^2e\cos\{(2-2m-c)\theta - 2\beta + \alpha\} \\ &\quad - \tfrac{15}{8}me^2\cos\{(2-2m-2c)\theta - 2\beta + 2\alpha\} \\ &\quad - \tfrac{3}{8}mk^2\cos\{(2-2m-2g)\theta - 2\beta + 2\gamma\}.\end{aligned}$$

Again

$$\begin{aligned}2\left(\frac{d^2u}{d\theta^2} + u\right)\int\frac{T}{h^2u^3}\,d\theta &= \tfrac{3}{2}m^2a\cos\{(2-2m)\theta - 2\beta\} \\ &\quad - 6m^2ae\cos\{(2-2m-c)\theta - 2\beta + \alpha\} \\ &\quad - \tfrac{15}{4}me^2a\cos\{(2-2m-2c)\theta - 2\beta + 2\alpha\} \\ &\quad - \tfrac{3}{4}mk^2a\cos\{(2-2m-2g)\theta - 2\beta + 2\gamma\}.\end{aligned}$$

The differential equation between u and θ will therefore become

$$\begin{aligned}\frac{d^2u}{d\theta^2} + u &= a[1 - \tfrac{3}{4}k^2 - \tfrac{1}{2}m^2 + \tfrac{3}{2}m^2e\cos(c\theta - \alpha) + \tfrac{3}{4}k^2\cos 2(g\theta - \gamma) \\ &- 3m^2\cos\{(2-2m)\theta - 2\beta\} + \tfrac{15}{2}m^2e\cos\{(2-2m-c)\theta - 2\beta + \alpha\} \\ &- \tfrac{3}{2}m^2e'\cos(m\theta + \beta - \zeta) + \tfrac{15}{4}me^2\cos\{(2-2m-2c)\theta - 2\beta + 2\alpha\} \\ &+ \tfrac{3}{4}mk^2\cos\{(2-2m-2g)\theta - 2\beta + 2\gamma\}].\end{aligned}$$

The integration of this equation gives

$$u = a\,[1 - \tfrac{3}{4}k^2 - \tfrac{1}{2}m^2 + e\cos(c\theta - \alpha) - \tfrac{1}{4}k^2\cos 2\,(g\theta - \gamma)$$
$$+ m^2\cos\{(2-2m)\,\theta - 2\beta\} + \tfrac{15}{8}me\cos\{(2-2m-c)\,\theta - 2\beta + \alpha\}$$
$$- \tfrac{3}{2}m^2e'\cos(m\theta + \beta - \zeta) + \tfrac{15}{4}me^2\cos\{(2-2m-2c)\,\theta - 2\beta + 2\alpha\}$$
$$+ \tfrac{3}{4}mk^2\cos\{(2-2m-2g)\,\theta - 2\beta + 2\gamma\}].$$

The last term of this expression for u does not present itself in Mr. Godfray's equation.

Adopting this value of u, we shall see that, important terms only being retained, $\dfrac{1}{hu^2}$ is equal to

$$\frac{1}{ha^2}\,[1 + \tfrac{3}{2}e^2 + \tfrac{3}{2}k^2 + m^2 - 2e\cos(c\theta - \alpha)$$
$$+ \tfrac{3}{2}e^2\cos 2\,(c\theta - \alpha) + \tfrac{1}{2}k^2\cos 2\,(g\theta - \gamma)$$
$$- 2m^2\cos\{(2-2m)\,\theta - 2\beta\} - \tfrac{15}{4}me\cos\{(2-2m-c)\,\theta - 2\beta + \alpha\}$$
$$+ 3m^2e'\cos(m\theta + \beta - \zeta) - \tfrac{15}{8}me^2\cos\{(2-2m-2c)\,\theta - 2\beta + 2\alpha\}$$
$$- \tfrac{3}{2}mk^2\cos\{(2-2m-2g)\,\theta - 2\beta + 2\gamma\}].$$

Substituting in the equation

$$\frac{dt}{d\theta} = \frac{1}{hu^2}\cdot\left(1 - \int\frac{T}{h^2u^3}\,d\theta\right)$$

the expressions for the factors of the right-hand member, we have, as far as important terms are concerned,

$$ha^2\,\frac{dt}{d\theta} = 1 + \tfrac{3}{2}e^2 + \tfrac{3}{2}k^2 + m^2 - 2e\cos(c\theta - \alpha)$$
$$+ \tfrac{3}{2}e^2\cos 2\,(c\theta - \alpha) + \tfrac{1}{2}k^2\cos 2\,(g\theta - \gamma)$$
$$- \tfrac{11}{4}m^2\cos\{(2-2m)\,\theta - 2\beta\} - \tfrac{15}{4}me\cos\{(2-2m-c)\,\theta - 2\beta + \alpha\}$$
$$+ 3m^2e'\cos(m\theta + \beta - \zeta) - \tfrac{9}{8}mk^2\cos\{(2-2m-2g)\,\theta - 2\beta + 2\gamma\},$$

whence

$$pt = \theta - 2e\sin(c\theta - \alpha) + \tfrac{3}{4}e^2\sin 2\,(c\theta - \alpha) + \tfrac{1}{4}k^2\sin 2\,(g\theta - \gamma)$$
$$- \tfrac{11}{8}m^2\sin\{(2-2m)\,\theta - 2\beta\} - \tfrac{15}{4}me\sin\{(2-2m-c)\,\theta - 2\beta + \alpha\}$$
$$+ 3me'\sin(m\theta + \beta - \zeta) + \tfrac{9}{16}k^2\sin\{(2-2m-2g)\,\theta - 2\beta + 2\gamma\}.$$

In the *Treatises on the Lunar Theory* by Mr. Airy and Archdeacon Pratt, the investigations leading to the determination of the true longitude in terms of the mean are conducted less minutely than by Mr. Godfray, inasmuch as the

term involving e^2 in the approximate value of $\frac{T}{h^2u^3}$ is neglected in their researches. Since the terms, of which the argument is $2-2m-2c$, which result from the expressions obtained by Mr. Godfray for

$$\frac{1}{hu^2} \text{ and } 1-\int\frac{T}{h^2u^3}\,d\theta,$$

cancel each other in the expression for $\frac{dt}{d\theta}$, Mr. Airy's and Archdeadon Pratt's investigations terminate in the same result as Mr. Godfray's.

All these writers in their several treatises neglect the term

$$\frac{k^2}{a^4}\cos 2\,(g\theta-\gamma)$$

in the expression for $\frac{1}{u^4}$: yet this term seems to have as good a title to our notice as the term

$$\frac{10e^2}{a^4}\cos^2(c\theta-\alpha),$$

which is retained by Mr. Godfray.

If the term, of which $2\,(g\theta-\gamma)$ is the argument, be retained, then, as I have shewn, the above process leads to the following term in the expression for the mean longitude, viz.

$$\tfrac{9}{16}k^2\sin\{(2-2m-2g)\,\theta-2\beta+2\gamma\}.$$

On referring to Plana's *Théorie du Mouvement de la Lune*, tom. I., p. 491, it will be seen that, in the expression for the time in terms of the true longitude, we have the term

$$(0\,.\,k^2+\&c.)\sin\{(2-2m-2g)\,\theta-2\beta+2\gamma\}.$$

The state of the question seems therefore to be as follows: pt contains the terms

$$0\,.\,e^2\sin\{(2-2m-2c)\,\theta-2\beta+2\alpha\},$$

$$0\,.\,k^2\sin\{(2-2m-2g)\,\theta-2\beta+2\gamma\}.$$

Airy and Pratt ignore these terms: Godfray calculates the term $0\,.\,e^2$, but he ignores the term $0\,.\,k^2$. If for the terms in k^2 he had carried the investigation as far as in the corresponding investigation was sufficient to give him $0\,.\,e^2$, he would have obtained the term $\frac{9}{16}k^2$. This term must there-

fore be destroyed by an opposite term $-\frac{9}{16}k^2$ introduced by a later approximation.

The investigations given in this note point out that the results obtained in our ordinary treatises for the place of the moon at any time, although in fact true as far as the second order of small quantities, are not in these treatises conclusively proved to be true even to this order. Recourse to higher approximations than those which are adopted in these works appears to be necessary in order to fix conclusively the coordinates of the moon's place even to the second order of small quantities.

August 31, 1866.

CRITICAL EXAMINATION OF EUCLID'S FIRST PRINCIPLES COMPARED TO THOSE OF MODERN GEOMETRY, ANCIENT AND MODERN ANALYSIS.

By Ferdinand Wolff.

(Continued from Vol. VII. p. 81.)

WE have seen by a former article that in applying to I. 26, Pappus' principle of inverted superposition, I. 6 becomes a corollary to I. 26, just as I. 5 is to I. 4.

These two propositions in their more general form are thus enunciated.

1. If two triangles have two sides of the one equal to two sides of the other, not only each to each, but each to either, and have also the angles contained by the sides equal, the remaining angles shall be also equal, not only each to each, but each to either.

Otherwise: In an isosceles triangle the angles upon the base are equal to each other.

2. If two triangles have two angles of the one equal to two angles of the other, not only each to each, but each to either, and have also the side adjacent to the equal angles equal, then shall the other sides be equal, not only each to each, but each to either.

Otherwise: If two angles of a triangle be equal to each other, the sides which subtend the equal angles shall be equal.

Pappus' principle of inverted superposition is in fact nothing else than the same principle applied by Euclid himself in a more general way for the demonstration of I. 5.

Applying the triangle *DEF* (figs. 19, 20, 21) inversedly upon *ABC*, and joining *FC*, we form the isosceles triangle *AFC* and have Euclid's demonstration. Joining *BE*, we form the isosceles triangle *ABE* and get Proclus' demonstration, and, finally, if in applying the two triangles inversedly one upon the other, *DF* be equal to *AB* and *ED* to *AC*, we come to Pappus' demonstration; that is, we apply the isosceles triangle inversedly upon itself.

The eighth proposition establishes the coincidence of triangles having their three sides respectively equal. As the demonstration of this proposition based upon I. 7 involves a principle more proper than any other to show the distinctive character of Ancient and Modern Geometry, we have to refute first the theory of those writers who attempt to supersede in this case "superposition" by "apposition."

"I. 7," says Mr. Todhunter, "is only required to lead to I. 8." The two might be superseded by another demonstration of I. 8, which has been recommended by many writers.

"Let *ABC*, *DEF* (figs. 22, 23, 24) be two triangles having the sides *AB* and *AC* equal to *DE*, *DF*, each to each, and the base *BC* equal to the base *EF*: the angle *BAC* shall be equal to the angle *EDF*. For, let the triangle *DEF* be applied to the triangle *ABC*, so that the sides may coincide, the equal sides be conterminous, and the vertices fall on opposite sides of the base. Let *GBC* represent the triangle *DEF* thus applied, so that *G* corresponds to *D*. Join *AG*. Since, by hypothesis, *BA* is equal to *BG*, the angle *BAG* is equal to the angle *BGA*, by I. 5. In the same manner the angle *CAG* is equal to the angle *CGA*. Therefore the whole angle *BAC* is equal to the whole angle *BGC*; that is, the angle *EDF*, &c." (Todhunter's *Euclid*, p. 256).

If the triangle *BGC* represents the triangle *DEF*, it must also represent any other triangle as *XYZ*, having its three sides equal to the three sides of *ABC*, each to each. For what I. 8 has to prove is that a triangle is given, determined by its three sides being given, and that any other triangle having its three sides respectively equal to those of the given one shall coincide with it. Now if we admit that *DEF*, as well as *XYZ*, is represented by and coincident with the triangle *BGC*, it follows that *DEF* must also coincide with *XYZ*. But that is just what has to

be proved; all we know of DEF and XYZ is that the three sides of the one are equal to the three sides of the other. Thus, instead of proving the proposition, Mr. Todhunter only shifts the question, he moves in what we call a vicious circle (*il tourne dans un cercle vicieux*).

But without going so far, how are we to appose the triangle DEF on the triangle ABC, so as to make it equal to the triangle BGC? To make the angles BCG or CBG equal to DFE or DEF, we want the problem I. 23, and the demonstration of this construction reposes precisely upon I. 8. Nor are we a step farther advanced by making the sides CG and BG equal to DF and DE; as we only form a triangle, the three sides of which are equal to those of DEF, and then it still remains to be proved that the triangle BGC shall be coincident with EDF, what Todhunter at the very outset admits without proving it.

Returning to I. 7 it is obvious that this proposition properly extended is identical with III. 10: two circles cannot cut in more than two points.

In the original text this proposition is thus enunciated: Upon the same straight line, and at two different points situated on the same side of it, there cannot be drawn two lines equal to two other lines, and having the same extremities as the two first lines.

Taking the two points of the original text, instead of Robert Simson's interpretation, the locus for this impossible point D (fig. 25), if $DB=BC$, is a circle described with B as centre and BC as radius; and the locus for the impossible point D', if $AD=AC$, is a circle described with A as centre and AC as radius.

The affinity between the first, the eighth, and the twenty-second proposition is at once evident.

In the eighth proposition Euclid proves that there is only *one* triangle possible with three given sides, and the locus for the vertices of all the other impossible triangles is given by the two circles.

In the twenty-second proposition a triangle "with three sides on the same base and on the same side of," &c. is to be described; but the limitation is given, under which the construction is possible; this limitation is that two sides must be greater than the third.

In the first proposition the triangle to be described is to have its three sides equal.

Here then again Euclid, just as for I. 6, gives the special case before the general.

The general case is twenty-two; and in order to describe a triangle with three given sides, Euclid has first to prove by I. 8 that there can only one triangle be described, *i.e.* that the two circles can only meet in one point; (that is, on the same side of the base; that the circles meet on the other side of the base and form another triangle is quite out of consideration, although this very point has led ancient Philo, as well as modern writers, Mr. Todhunter included, to their erroneous demonstration). Then for the same proposition I. 22 Euclid had to establish by the twentieth proposition that the two sides must be greater than the third. And in Euclid's very first proposition, in the special case where the three sides of the triangle to be constructed shall be equal, those two points are premised without proof. But even this latter point is of minor importance and has been explained more or less satisfactorily by other writers.

What we have foremost to consider is the demonstration of I. 7 itself. According to the enunciation ABC is the only possible triangle upon the base AB, with AC and BC given as corresponding sides. For supposing there is another triangle possible with D (fig. 26) as vertex and $CB=DB$, and $AC=AD$. Then Euclid's demonstration amounts to this: if CBD is an isosceles triangle, ACD cannot be one. In other words, the general proposition of I. 7 is reduced to the more special one: there cannot be on the same base CD two isosceles triangles, having their vertices situated on two different lines (lines of reflection of C and D). And this latter proposition again is reduced to the still more special ones.

There cannot be on the same base two equilateral triangles having their vertices on two different points, (on the same side of their basis).

This is precisely the first proposition; and all that has been proved in I. 1 is that all the sides are alike, but it has not been proved that the circles meet only in one point. Now I. 2 and I. 3, the sequel of I. 1, is indispensable for the demonstration of I. 5; for Euclid, in order to show that the lines to be cut off or to be made equal to other lines are all situated in the same rectilineal plane, has to draw them from common centres, or from centres situated on lines which form the rectilineal plane. And for that purpose he constructs first his equilateral triangle as the common rectilineal plane for all the other figures or lines.

Thus I. 5, the demonstration of which requires I. 1 and its sequel I. 3, to which Euclid refers in his demonstration,

is required in its turn to demonstrate I. 7. This latter proposition is, in fact as we have seen, nothing else than I. 5 differently enunciated. Instead of saying, that in an isosceles triangle the angles on the base are equal; Euclid says, there cannot be two isosceles triangles on the same base CD, having their vertices on two different straight lines (the lines of reflection of the base), which proposition is closely connected with I. 1, or its sequel I. 3.

En Resumé.

The demonstration of the general case that two triangles are equal when the three sides of the one are equal to the three sides of the other, each to each, is based upon the particular case that two triangles are equal when the three sides of the one are equal to the three sides of the other, not only each to each, but each to either; that is, when the two triangles are equilateral and the demonstration of this proposition has not been given.

More generally: by I. 7 Euclid demonstrates that two circles intersect only in two points, and the demonstration of this general proposition is based upon the special case, that two circles described with equal radii and from two centres, the distance of which is equal to the radius, intersect only in two points, which latter proposition, the first in Euclid, is therefore to be considered as an axiom.

In opposition to the ancient Geometry, which thus begins with the most special case of equal circles, intersecting in two points only; the Modern Geometry begins by asserting that all the circles in the same plane intersect at the *same two points* at infinity, and derives from it the reason why two circles in the same plane cannot cut each other in more than two points.

Of all the different data in the different propositions, the datum constantly to be referred to as common to all in planometry is that of the common plane.

How does Euclid satisfy this condition? By the equilateral triangle. For instance, in I. 3, "to cut off from a given straight line a line equal to another given straight line," he requires for the complete construction no less than five circles. Joining XA (fig. 27), he wants first two circles to construct the equilateral triangle, then he requires three more to bring AB successively upon XY, thus $AB = AN = XL = XS$, all the circles are drawn as we see from the angles of the equilateral triangle.

The demonstration of I. 4 requires five, if not seven circles. Two circles are required for the construction of an equilateral triangle upon the line joining the two vertices of the two given triangles. Three other circles are wanted to make $AB = EF$, while placing the triangle DEF upon ABC, and two more circles for placing EF upon BC. And all these circles are required to satisfy the condition of the common rectilineal plane, the ground-work of which is formed by the equilateral triangle. So much for Ancient Geometry.

Now how does Modern Geometry proceed?

We have two criteria for the rectilineal plane, one by straight lines and another by circles. First as to lines: we say the proposition is one of planometry and the condition of the plane is satisfied when all the straight lines in it intersect each other. If they do not visibly intersect, as is the case with parallel lines, they intersect in the mind's eye at infinity. If we want to investigate the properties which result from their different mode of intersection, we take, what Brianchon calls, a *bundle of lines*, un "*faisceau de lignes*" (pencil), intersecting in one point and intersected by other lines; and we obtain such a powerful method for investigating the pure relations of space that Steiner regards the theory of the pencil as the fundamental law governing the formation of all the geometrical figures. What for Brianchon was a "*bundle of lines*," "*un faisceau de lignes*," Steiner idealises in a "*strahlenbuschel*" (a set of rays).

As for circles we have already stated that, with regard to the same plane, they all intersect in the same two points at infinity. Thus if we have to investigate the properties of a "batch" of circles which do not really intersect, we take their radical axis as representative of their chords of intersection. What for the "*bundle of lines*" or pencil, was the common point of intersection, that is for the *batch* of circles, the straight line or the radical axis. Thus the pencil finds its counterpart in the system of coaxal circles with their common radical axis and limiting points.

We must not forget that, even in algebraical analysis, the Cartesian system is based upon the principle of referring all the lines and figures to the same plane, and this plane is formed by three lines; the third line lying at infinity as Professor Salmon ingeniously suggests. Therefore instead of the ordinary coordinates we have trilinear coordinates forming thus a triangle of reference as the common rectilineal plane.

Then again the relation between concurrent lines and collinear points, as manifested in the centre and axis of perspective, becomes therefore in Modern Geometry a source fertile in results for the investigation of figures situated in the same plane.

The datum common tò all propositions, as we have said, is in planometry that of the common plane; that is, of all the lines intersecting each other in one point, and of the circles intersecting or touching in two points, with the necessary regard to the intersection at infinity. In the one case we form the structure of a plane by a network of straight lines, and in the other by a ground-work of circles, both subjected to a common law. How the latter case can be reduced to the first case by putting the pencils of lines into what Steiner calls an "*ungleichliegende*" position, we need not dwell upon. It is important to remark here that the ancients did not limit themselves to the equilateral triangle for the structure of a plane. How anxious they were to form a plane upon a larger foundation, by forming a network of straight lines in order to grasp at once at all the points of the plane, and to exhaust it by straight lines, is most beautifully illustrated, first, by the famous Porism of Pappus, in his preface to Book VII. of *Collectiones Mathematicae;* and secondly, by Lemma XIII. to *Euclid's Porisms* also given by Pappus in the same book.

It is here that the Ancient Analysis borders closely on the domain of Modern Geometry. Proper reserve however must be made that the ancients, having been precluded from the idea of infinity and evanescence, were also precluded from the power of reducing variety of forms to uniformity and of deducing from uniformity the greatest variety.

As for the famous Porism which was first demonstrated by Robert Simson, the condition of the plane is satisfied by there being given any number of straight lines intersecting at random. This Porism has afterwards been reduced by Steiner to a simple proposition of Carnot's complete quadrilateral. But the complete quadrilateral itself has as basis an incomplete pencil, or rather a particular kind of pencil, the harmonical pencil. Carnot's laws of the harmonical pencil, known already in a great measure to the Ancients, were the forerunner to the more general form of pencil, or of Brianchon's anharmonical pencil.

What most distinguishes Ancient Geometry from the Modern is that the idea of infinity and evanescence was incompatible with the plastic mind of the Greeks. Limitation

was their first requirement. If you had told them that parallel lines meet at infinity, Euclid would have been the first to cry out, "By Pluto, do they meet then in the kingdom of shades, as they cannot meet in this world *though indefinitely produced.*" Euclid does not mean to say, *infinitely produced.* And as for circles if Poncelet himself had expounded in the face of Greece his theory of all the circles cutting in the same two points at infinity, Pythagoras, foremost amongst all the Greeks, would have pounced upon him exclaiming in despair, "At two points indeed, and if the one lies in the Elysian fields and the other in the infernal regions, by Cerberus how can I ever hope to get at them both?"

The mystic hexagon has at all times exercised a strange fascination over the mind of the greatest mathematicians; Poncelet amongst the French, Steiner and Plücker amongst the Germans, and Professor Cayley amongst the English, have made it the subject of their particular investigation. It appeared to them that to investigate the secret properties of the mystic hexagon was to reveal the fundamental laws of all geometrical forms and to overtake them as if it were in *flagrante delicto* of formation.

The same may be said of Pappus' Porism, which we mentioned before. Of this last Porism, the *Educational Times* has at intervals given variations of a more general character than the original theorem with elegant Demonstrations by Professor Cayley.

The process of generalisation has been carried to the highest pitch in the latter Numbers of the *Educational Times*, by the Problem called "The Four Points" Problem, where a random number of lines intersect at random and where solutions are to be given with respect to the random points of intersection. We dare not enter into the discussion still going on in the *Educational Times* lest we should come to random conclusions.

We must bear in mind that it was in the first instance Algebraical Analysis which led to the Geometrical Analysis of modern times. It was the very fixity of the algebraical formulæ which led to investigate the very mobility of geometrical forms from one fixed and general point of view to make *one* demonstration hold good for the most various forms of geometrical theorems.

What Leibnitz says about scientific terms being a bill drawn at sight upon our understanding applies with greater force to algebraical formulæ. The formulæ drawn, as it

were at sight upon our understanding have to be discounted in cash; that is, to be enunciated in words.

We have to give to the symbolic formulæ a tangible, comprehensible form.

It was no doubt by Algebraical Analysis that Newton was led to his method of generating conics by pencils, although he could not have known the pencil theorem. Thus his researches connect themselves with those of Brianchon and Chasles. True, as we said before, Newton could not know anything about the pencil theorem; the anharmonical pencil made its first appearance only in 1817, but what Newton did know was the importance of the geometrical interpretations of algebraical formulæ; nobody more than he regretted not having sooner or more assiduously applied himself to the study of Pure Geometry.

Steiner, on the contrary, anxious to blend Synthesis with Analysis, to conciliate Poncelet and Gergonne, and to give us Pure Geometry, purer than Euclid, could not declaim enough against those who study Euclid at the very outset.

"Give me a peasant boy," he used to say, "who has been all his lifetime whistling behind the plough without dreaming of Euclid, and I will teach him mathematics. But those students who come with their heads full of Euclidean demonstrations I have the greatest difficulty to teach"—my mathematics; which last words, though not spoken, were evidently meant.

Spring-Bank Terrace, Blackburn.

ON THE GENERALIZATION OF CERTAIN FORMULÆ INVESTIGATED BY MR. BLISSARD.

By Worontzof, M.M.

(Concluded from p. 208).

II. On the Summation of a Certain Series.

It is known that, if the result of the operation of the following symbolic series

$$\left\{f(\Delta, D)\,\frac{\Delta}{1} - f(\Delta, D)\,\frac{\Delta^2}{2} + f(\Delta, D)\,\frac{\Delta^3}{3} - \ldots\right\} \text{ upon } F(x),$$

for $x = n$ is a convergent series, then the finite value of $f(\Delta, D)\,DF(n)$ expresses (exactly) the sum of this series.

1. Since

$$D^v\Delta^m\Sigma_x = \{D^{v-1}\Delta^{m+1} - \tfrac{1}{2}D^{v-1}\Delta^{m+2} + \tfrac{1}{3}D^{v-1}\Delta^{m+3} - \ldots\}\,\Sigma_x, \ (\Delta x = 1),$$

$$D\sum_{x+1}^{x+\infty}\frac{1}{(x+1)^v} = \left\{\frac{\Delta}{1} - \frac{\Delta^2}{2} + \frac{\Delta^3}{3} - \ldots\right\}\sum_{x+1}^{x+\infty}\frac{1}{(x+1)^v},$$

where $\displaystyle\sum_{x+1}^{x+\infty}\frac{1}{(x+1)^v} = \frac{1}{(x+1)^v} + \frac{1}{(x+2)^v} + \frac{1}{(x+3)^v} + \ldots + \frac{1}{(x+\infty)^v}$,

$$D^v\Sigma_x = \{D^{v-1}\Delta - \tfrac{1}{2}D^{v-1}\Delta^2 + \ldots\}\,\Sigma_x,$$

$$D^v\Sigma_{x+m} = D^v\Sigma_x + mD^v\Delta\Sigma_x + \frac{m(m-1)}{1.2}\,D^v\Delta^2\Sigma_x + \ldots + D^v\Delta^m\Sigma_x,$$

and $$\frac{D^v}{\Delta}\,\Sigma_x = -\{D^v\Sigma_x + D^v\Sigma_{x+1} + D^v\Sigma_{x+2} + \ldots + \text{const.}\}$$
$$= \{D^{v-1} - \tfrac{1}{2}D^{v-1}\Delta + \tfrac{1}{3}D^{v-1}\Delta^2 - \ldots\}\,\Sigma_x,$$

supposing $x = n$,* we have

$$\frac{v}{m}\,\phi_v(n, m) = \frac{1}{n+m+1}\,\phi_{v-1}(n, m+1)$$
$$+ \frac{m+1}{(n+m+1)(n+m+2)}\,\frac{\phi_{v-1}(n, m+2)}{2}$$
$$+ \frac{(m+1)(m+2)}{(n+m+1)(n+m+2)(n+m+3)}\,\frac{\phi_{v-1}(n, m+3)}{3} + \ldots \ \ldots\text{(I)},$$

* n, m, v positive integers, $v > 0$.

$$\frac{v}{\Gamma(n+1)}\left\{\frac{1}{(n+1)^{v+1}}+\frac{1}{(n+2)^{v+1}}+\frac{1}{(n+3)^{v+1}}+\dots\right\}$$

$$=\frac{v}{\Gamma(n+1)}\left\{f_{v+1}-\Sigma_n(v+1)\right\}$$

$$=\frac{\phi_{v-1}(n,1)}{1.1.2.3\dots(n+1)}+\frac{\phi_{v-1}(n,2)}{2.2.3.4\dots(n+2)}+\frac{\phi_{v-1}(n,3)}{3.3.4.5\dots(n+3)}+\dots$$

$$=\frac{(-1)^{v-1}}{\Gamma v\,\Gamma(n+1)}D^v\Sigma_n \;\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\text{(II)},$$

$$D^v\Sigma_{n+m}-D^v\Sigma_n=m\Gamma(v+1)(-1)^v\left\{\frac{\phi_v(n,1)}{n+1}-\frac{m-1}{2}\,\frac{\phi_v(n,2)}{(n+1)(n+2)}\right.$$

$$+\frac{(m-1)(m-2)}{3}\,\frac{\phi_v(n,3)}{(n+1)(n+2)(n+3)}+\dots$$

$$\left.+(-)^{m-1}\frac{(m-1)(m-2)\dots2.1}{m}\,\frac{\phi_v(n,m)}{(n+1)(n+2)\dots(n+m)}\right\}\dots\text{(III)},$$

or $D^v\Sigma_{n+m}-D^v\Sigma_n=(-)^v\Gamma(v+1)\left\{\Sigma_{n+m}(v+1)-\Sigma_n(v+1)\right\}$

$$\dots\dots\dots\dots\dots\dots\text{(IV)},$$

and $\{D^v\Sigma_n+D^v\Sigma_{n+1}+D^v\Sigma_{n+2}+\dots\}-\{D^v\Sigma_m+D^v\Sigma_{m+1}+D^v\Sigma_{m+2}+\dots\}$

$$=D^{v-1}\Sigma_m-D^{v-1}\Sigma_n$$

$$+(-)^{v-1}\Gamma v\left[\Gamma(n+1)\left\{\frac{\phi_{v-1}(n,1)}{2.1.2.3\dots(n+1)}+\frac{\phi_{v-1}(n,2)}{3.2.3.4\dots(n+2)}+\dots\right\}\right.$$

$$\left.-\Gamma(m+1)\left\{\frac{\phi_{v-1}(m,1)}{2.1.2.3\dots(m+1)}+\frac{\phi_{v-1}(m,2)}{3.2.3.4\dots(m+2)}+\dots\right\}\right],$$

or, in the last formula, putting $m+n$ for m,

$$D^v\Sigma_n+D^v\Sigma_{n+1}+D^v\Sigma_{n+2}+\dots+D^v\Sigma_{n+m-1}=D^{v-1}\Sigma_{n+m}-D^{v-1}\Sigma_n$$

$$+(-1)^{v-1}\Gamma v\left[\Gamma(n+1)\left\{\frac{\phi_{v-1}(n,1)}{2.1.2.3\dots(n+1)}+\frac{\phi_{v-1}(n,2)}{3.2.3.4\dots(n+2)}\right.\right.$$

$$\left.+\frac{\phi_{v-1}(n,3)}{4.3.4.5\dots(n+3)}+\dots\right\}$$

$$\left.-\Gamma(m+n+1)\left\{\frac{\phi_{v-1}(m+n,1)}{2.1.2.3\dots(m+n+1)}+\frac{\phi_{v-1}(m+n,2)}{3.2.3.4\dots(m+n+2)}+\dots\right\}\right]$$

$$\dots\dots\dots\dots\dots\text{(V)}.$$

Now substituting in (V) for $D^v\Sigma_n$, $D^v\Sigma_{n+1}$, ... their values from (II), we get

$$\Sigma_{n+m}(v)-\Sigma_n(v)-v[m\,f_{v+1}-\{\Sigma_n(v+1)+\Sigma_{n+1}(v+1)+\dots+\Sigma_{n+m-1}(v+1)\}]$$

$$=\Gamma(m+n+1)\left\{\frac{\phi_{v-1}(m+n,\,1)}{2.1.2.3\ldots(m+n+1)}+\frac{\phi_{v-1}(m+n,\,2)}{3.2.3.4\ldots(m+n+2)}+\frac{\phi_{v-1}(m+n,\,3)}{4.3.4.5\ldots(m+n+3)}+\ldots\right\}$$

$$-\Gamma(n+1)\left\{\frac{\phi_{v-1}(n,\,1)}{2.1.2.3\ldots(n+1)}+\frac{\phi_{v-1}(n,\,2)}{3.2.3.4\ldots(n+2)}+\frac{\phi_{v-1}(n,\,3)}{4.3.4.5\ldots(n+3)}+\ldots\right\}\quad\ldots\ldots\ldots\ldots\ldots\ldots\text{(VI)}.$$

Ex. From (I)

$$(v=1,\,m=1)\;\frac{1}{(n+1)\,\Gamma(n+2)}=\frac{1}{1.2.3\ldots(n+2)}+\frac{1}{2.3.4\ldots(n+3)}+\ldots\quad\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

$$(v=1,\,n=0)\;\frac{\Sigma_m}{m}=\frac{1}{1\,(m+1)}+\frac{1}{2\,(m+2)}+\frac{1}{3\,(m+3)}+\ldots\quad\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

$$(v=2,\,m=1)\;\frac{1}{(n+1)\,\Gamma(n+2)}\left\{\frac{2}{n+1}+\Sigma_n\right\}$$

$$=\frac{\Sigma_{n+2}}{1.2.3\ldots(n+2)}+\frac{\Sigma_{n+3}}{2.3.4\ldots(n+3)}+\ldots\;\ldots\ldots(3),$$

$$(v=2,\,n=0)\;\frac{2}{m}\,\phi_2(0,\,m)=\frac{2}{m}\left(\frac{\Sigma_1}{1}+\frac{\Sigma_2}{2}+\ldots+\frac{\Sigma_m}{m}\right)$$

$$=\frac{\Sigma_{m+1}}{1\,(m+1)}+\frac{\Sigma_{m+2}}{2\,(m+2)}+\frac{\Sigma_{m+3}}{3\,(m+3)}+\ldots\;\ldots\ldots(4).$$

But $\phi_2(n,\,m)=\frac{1}{2}\{(\Sigma_{n+m}-\Sigma_n)^2+\Sigma_{n+m}(2)-\Sigma_n(2)\}$;
therefore

$$\frac{\Sigma_m(2)+(\Sigma_m)^2}{m}=\frac{\Sigma_{m+1}}{1(m+1)}+\frac{\Sigma_{m+2}}{2(m+2)}+\frac{\Sigma_{m+3}}{3(m+3)}+\ldots\;\ldots(5).$$

Hence
$$\frac{\Sigma_2}{1.2}+\frac{\Sigma_3}{2.3}+\frac{\Sigma_4}{3.4}+\ldots=2,$$

$$\frac{\Sigma_3}{1.3}+\frac{\Sigma_4}{2.4}+\frac{\Sigma_5}{3.5}+\ldots=\tfrac{7}{4},$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{ etc.},$$

$(v=3,\,m=1)$

$$\frac{1}{(n+1)\,\Gamma(n+2)}\left\{\frac{6}{(n+1)^2}+2\Sigma_n\left(\frac{2}{n+1}+\Sigma_n\right)+\Sigma_n(2)-(\Sigma_n)^2\right\}$$

$$=\frac{\Sigma_{n+2}(2)+(\Sigma_{n+2})^2}{1.2.3\ldots(n+2)}+\frac{\Sigma_{n+3}(2)+(\Sigma_{n+3})^2}{2.3.4\ldots(n+3)}+\ldots\;\ldots\ldots(6),$$

$$(v=3,\ n=0)\ \frac{6}{m}\,\phi_3(0,\ m)$$

$$=\frac{\Sigma_{m+1}(2)+(\Sigma_{m+1})^2}{1\,(m+1)}+\frac{\Sigma_{m+2}(2)+(\Sigma_{m+2})^2}{2\,(m+2)}+\ldots\ \ldots\ldots(7).$$

Hence $\dfrac{\Sigma_2(2)+(\Sigma_2)^2}{1.2}+\dfrac{\Sigma_3(2)+(\Sigma_3)^2}{2.3}+\dfrac{\Sigma_4(2)+(\Sigma_4)^2}{3.4}+\ldots=6,$

.... etc.,

$$(v=v,\ m=1)\ \frac{v}{(n+1)^v\,\Gamma(n+1)}$$

$$=\frac{\phi_{v-1}(n,\ 2)}{1.2.3\ldots(n+2)}+\frac{\phi_{v-1}(n,\ 3)}{2.3.4\ldots(n+3)}+\ldots\ \ldots\ldots(8).$$

Hence, if $n=0$,

$$v=\frac{\phi_{v-1}(0,\ 2)}{1.2}+\frac{\phi_{v-1}(0,\ 3)}{2.3}+\frac{\phi_{v-1}(0,\ 4)}{3.4}+\ldots\ \ldots\ldots(9).$$

From (II)

$$(v=1)\ \frac{1}{\Gamma(n+1)}\,D\Sigma_n=\frac{1}{\Gamma(n+1)}\left\{\frac{\pi^2}{6}-\Sigma_n(2)\right\}$$

$$=\frac{1}{1.1.2.3\ldots(n+1)}+\frac{1}{2.2.3.4\ldots(n+2)}+\frac{1}{3.3.4.5\ldots(n+3)}+\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots(10).$$

Hence $$D\Sigma_0=\frac{\pi}{6}=\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\ldots,$$

$$D\Sigma_1=\frac{\pi^2}{6}-1=\frac{1}{1^2 2}+\frac{1}{2^2 3}+\frac{1}{3^2 4}+\ldots,$$

$$\frac{1}{\Gamma 3}\,D\Sigma_2=\frac{1}{\Gamma 3}\left\{\frac{\pi^2}{6}-\Sigma_2(2)\right\}=\frac{1}{1^2.2.3}+\frac{1}{2^2.3.4}+\frac{1}{3^2.4.5}+\ldots,$$

.... etc.,

$$(v=2)\ \frac{-1}{\Gamma(n+1)}\,D^2\Sigma_n=\frac{2}{\Gamma(n+1)}\,\{\textstyle\int_3-\Sigma_n(3)\}$$

$$=-\frac{\Sigma_n}{\Gamma(n+1)}\left\{\frac{\pi^2}{6}-\Sigma_n(2)\right\}$$

$$+\frac{\Sigma_{n+1}}{1.1.2.3\ldots(n+1)}+\frac{\Sigma_{n+2}}{2.2.3.4\ldots(n+2)}+\frac{\Sigma_{n+3}}{3.3.4.5\ldots(n+3)}+\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots(11).$$

Hence

$$-D^2\Sigma_0 = 2\int_3 = \frac{\Sigma_1}{1^2} + \frac{\Sigma_2}{2^2} + \frac{\Sigma_3}{3^2} + \frac{\Sigma_4}{4^2} + \ldots$$

$$-D^2\Sigma_1 = 2(\int_3 - 1) = -\left(\frac{\pi^2}{6} - 1\right) + \frac{\Sigma_2}{1^2.2} + \frac{\Sigma_3}{2^2.3} + \frac{\Sigma_4}{3^2.4} + \ldots$$

$$\frac{-1}{\Gamma 3} D^2\Sigma_2 = \frac{2}{\Gamma 3}\{\int_3 - \Sigma_2(3)\}$$

$$= -\frac{\Sigma_2}{\Gamma 3}\left\{\frac{\pi^2}{6} - \Sigma_2(2)\right\} + \frac{\Sigma_3}{1^2.2.3} + \frac{\Sigma_4}{2^2.3.4} + \ldots$$

.... etc.,

and, generally,

$$\frac{\Sigma_{n+1}}{1.1.2.3\ldots(n+1)} + \frac{\Sigma_{n+2}}{2.2.3.4\ldots(n+2)} + \ldots$$

$$= \frac{1}{\Gamma(n+1)}\left[2\{\int_3 - \Sigma_n(3)\} + \Sigma_n\left\{\frac{\pi^2}{6} - \Sigma_n(2)\right\}\right] \ldots\ldots (12).$$

From (VI)

$$(v=1,\ n=0)\ 1 - m\left\{\frac{\pi^2}{6} - \Sigma_m(2)\right\}$$

$$= \Gamma(m+1)\left\{\frac{1}{2.1.2.3\ldots(m+1)} + \frac{1}{3.2.3.4\ldots(m+2)} + \frac{1}{4.3.4.5\ldots(m+3)} + \ldots\right\} \ldots\ldots\ldots (13).$$

Hence
$$2 - \frac{\pi^2}{6} = \frac{1}{1.2^2} + \frac{1}{2.3^2} + \frac{1}{3.4^2} + \ldots,$$

$$\frac{7}{4} - \frac{\pi^2}{6} = \frac{1}{1.2^2.3} + \frac{1}{2.3^2.4} + \frac{1}{3.4^2.5} + \ldots,$$

.... etc.,

$$(v=2,\ n=0)\ \Sigma_m\left[1 - m\left\{\frac{\pi^2}{6} - \Sigma_m(2)\right\}\right]$$

$$+\Sigma_m(2) - 2[m\int_3 - \{\Sigma_1(3) + \Sigma_2(3) + \ldots + \Sigma_{m-1}(3)\}] + \frac{\Sigma_1}{1.2} + \frac{\Sigma_2}{2.3} + \frac{\Sigma_3}{3.4} + \ldots$$

$$= \Gamma(m+1)\left\{\frac{\Sigma_{m+1}}{2.1.2.3\ldots(m+1)} + \frac{\Sigma_{m+2}}{3.2.3.4\ldots(m+2)} + \frac{\Sigma_{m+3}}{4.3.4.5\ldots(m+3)} + \ldots\right\}.$$

But $$\frac{\Sigma_1}{1.2}+\frac{\Sigma_2}{2.3}+\frac{\Sigma_3}{3.4}+\ldots=\frac{\pi^2}{6};$$

therefore $$\Sigma_m\left[1-m\left\{\frac{\pi^2}{6}-\Sigma_m(2)\right\}\right]$$

$$+\Sigma_m(2)+\frac{\pi^2}{6}-2\,[m\textstyle\int_3-\{\Sigma_1(3)+\Sigma_2(3)+\ldots+\Sigma_{m-1}(3)\}]$$

$$=\Gamma(m+1)\left\{\frac{\Sigma_{m+1}}{2.1.2.3\ldots(m+1)}+\frac{\Sigma_{m+2}}{3.2.3.4\ldots(m+2)}\right.$$

$$\left.+\frac{\Sigma_{m+3}}{4.3.4.5\ldots(m+3)}+\ldots\right\}\ldots\ldots\ldots(14).$$

Hence $$3-2\textstyle\int_3=\dfrac{\Sigma_2}{1.2^2}+\dfrac{\Sigma_3}{2.3^2}+\dfrac{\Sigma_4}{3.4^2}+\ldots,$$

$$\frac{34}{8}-\frac{\pi^2}{6}-2\textstyle\int_3=\dfrac{\Sigma_3}{1.2^2.3}+\dfrac{\Sigma_4}{2.3^2.4}+\dfrac{\Sigma_5}{3.4^2.5}+\ldots,$$

.... etc.

2. Let each member of the symbolic formulæ

$$D=\frac{\Delta}{1}-\frac{\Delta^2}{2}+\frac{\Delta^3}{3}-\ldots(\Delta x=1),$$

and $$\frac{D}{\Delta}=1-\frac{\Delta}{2}+\frac{\Delta^2}{3}-\ldots$$

operate on $\dfrac{\Sigma_{x+1}}{(x+1)^v}$ and $\dfrac{\Sigma_x}{(x+1)^v}$, and we obtain, supposing $x=n$, the following summations:

$$\frac{1}{(n+1)^v}\left\{v\,\frac{\Sigma_{n+1}}{n+1}-\frac{\pi^2}{6}+\Sigma_{n+1}(2)\right\}$$

$$=\Gamma(n+1)\left\{\frac{\theta_v(n,2)}{1.2.3\ldots(n+2)}+\frac{\theta_v(n,3)}{2.3.4\ldots(n+3)}+\frac{\theta_v(n,4)}{3.4.5\ldots(n+4)}+\ldots\right\}$$

..................(VII),

$$\frac{\pi^2}{6}-\Sigma_1(2)+\frac{1}{2^v}\left\{\frac{\pi^2}{6}-\Sigma_2(2)\right\}+\frac{1}{3^v}\left\{\frac{\pi^2}{6}-\Sigma_3(2)\right\}+\ldots$$

$$+\frac{1}{n^v}\left\{\frac{\pi^2}{6}-\Sigma_n(2)\right\}-v\left\{\frac{\Sigma_1}{1^{v+1}}+\frac{\Sigma_2}{2^{v+1}}+\ldots+\frac{\Sigma_n}{n^{v+1}}\right\}$$

$$=\Gamma(n+1)\left\{\frac{\theta_v(n,1)}{1.1.2.3\ldots(n+1)}+\frac{\theta_v(n,2)}{2.2.3.4\ldots(n+2)}+\frac{\theta_v(n,3)}{3.3.4.5\ldots(n+3)}+\ldots\right\}$$

$$-\left\{\frac{\theta_v(0,1)}{1^2}+\frac{\theta_v(0,2)}{2^2}+\frac{\theta_v(0,3)}{3^2}+\ldots\right\}\ldots\ldots\ldots(\text{VIII}),$$

$$\frac{1}{(n+1)^v}\left\{v\frac{\Sigma_n}{n+1}-\frac{\pi^2}{6}+\Sigma_n(2)\right\}$$

$$=\Gamma(n+1)\left\{\frac{\theta_v'(n,2)}{1.2.3\ldots(n+2)}+\frac{\theta_v'(n,3)}{2.3.4\ldots(n+3)}+\frac{\theta_v'(n,4)}{3.4.5\ldots(n+4)}+\ldots\right\}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots(\text{IX}),$$

$$\frac{\pi^2}{6}+\frac{1}{2^v}\left\{\frac{\pi^2}{6}-\Sigma_1(2)\right\}+\frac{1}{3^v}\left\{\frac{\pi^2}{6}-\Sigma_2(2)\right\}+\ldots$$

$$+\frac{1}{n^v}\left\{\frac{\pi^2}{6}-\Sigma_{n-1}(2)\right\}-v\left\{\frac{\Sigma_1}{2^{v+1}}+\frac{\Sigma_2}{3^{v+1}}+\ldots+\frac{\Sigma_{n-1}}{n^{v+1}}\right\}$$

$$=\Gamma(n+1)\left\{\frac{\theta_v'(n,1)}{1.1.2.3\ldots(n+1)}+\frac{\theta_v'(n,2)}{2.2.3.4\ldots(n+2)}+\frac{\theta_v'(n,3)}{3.3.4.5.6\ldots(n+3)}+\ldots\right\}$$

$$-\left\{\frac{\theta_v'(0,1)}{1^2}+\frac{\theta_v'(0,2)}{2^2}+\frac{\theta_v'(0,3)}{3^2}+\ldots\right\}\ldots\ldots\ldots(\text{X}).$$

Ex. From (VII)

$$(v=1)\ \frac{1}{(n+1)\,\Gamma(n+1)}\left\{\frac{\pi^2}{6}-\Sigma_n(2)\right\}$$

$$=\frac{\Sigma_1}{1.2.3\ldots(n+2)}+\frac{\Sigma_2}{2.3.4\ldots(n+3)}+\frac{\Sigma_3}{3.4.5\ldots(n+4)}+\ldots\ \ldots(15).$$

Hence
$$\frac{\Sigma_1}{1.2}+\frac{\Sigma_2}{2.3}+\frac{\Sigma_3}{3.4}+\ldots=\frac{\pi^2}{6},$$

$$\frac{\Sigma_1}{1.2.3}+\frac{\Sigma_2}{2.3.4}+\frac{\Sigma_3}{3.4.5}+\ldots=\frac{\pi^2}{1.2}-\tfrac{1}{2},$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{etc.},$$

$$(v=2,\ n=0)\ \frac{\Sigma_2(2)}{1.2}+\frac{\Sigma_3(2)}{2.3}+\frac{\Sigma_4(2)}{3.4}+\ldots=3-\frac{\pi^2}{6}\text{*}\ \ldots(16).$$

From (VIII)

$$(v=1)\ \frac{1}{\Gamma(n+1)}\Big[f_3-2\Sigma_n(3)-\Sigma_n\Sigma_n(2)$$

$$+\left\{\frac{\Sigma_1(2)}{1}+\frac{\Sigma_2(2)}{2}+\ldots+\frac{\Sigma_n(2)}{n}\right\}+\left\{\frac{\Sigma_1}{1^2}+\frac{\Sigma_2}{2^2}+\ldots+\frac{\Sigma_n}{n^2}\right\}\Big]$$

$$=\frac{\Sigma_1}{2.2.3.4\ldots(n+2)}+\frac{\Sigma_2}{3.3.4.5\ldots(n+3)}+\frac{\Sigma_3}{4.4.5.6\ldots(n+4)}+\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots(17).$$

* From the formula

$$D\frac{\Sigma_x}{(x+1)(x+2)}=\left(\frac{\Delta}{1}-\frac{\Delta^2}{2}+\ldots\right)\frac{\Sigma_x}{(x+1)(x+2)}\ x=0.$$

Hence $$\frac{\Sigma_1}{2^2} + \frac{\Sigma_2}{3^2} + \frac{\Sigma_3}{4^2} + \ldots = \int_3,$$

$$\frac{\Sigma_1}{2^2.3} + \frac{\Sigma_2}{3^2.4} + \frac{\Sigma_3}{4^2.5} + \ldots = \int_3 - 1,$$

....

Cor. The formula

$$D\Delta^n \frac{\Sigma_x}{x+1} = \left\{\frac{\Delta^{n+1}}{1} - \frac{\Delta^{n+2}}{2} + \ldots\right\} \frac{\Sigma_x}{x+1}$$

gives, supposing $x = 0$,

$$\frac{\Sigma_{n+1}}{1(n+2)} + \frac{\Sigma_{n+2}}{2(n+3)} + \frac{\Sigma_{n+3}}{3(n+4)} + \ldots = \frac{1}{n+1}\left\{\frac{\pi^2}{6} + \Sigma_n \Sigma_{n+1}\right\}$$

..................(18).

Hence $$\frac{\Sigma_2}{1.3} + \frac{\Sigma_3}{2.4} + \frac{\Sigma_4}{3.5} + \ldots = \frac{\pi^2}{1.2} + \frac{3}{4},$$

.... etc.

Also $$\frac{\Sigma_1}{1.3} + \frac{\Sigma_2}{2.4} + \frac{\Sigma_3}{3.5} + \ldots = \frac{\pi^2}{1.2} + \frac{1}{2}.$$

3. Several other series of the like kind may be easily summed by the aid of the following symbolic formulæ:

$$D(1+\Delta)^n - \Sigma_n (1+\Delta)^n + \frac{n}{1}\,\frac{(1+\Delta)^{n-1}}{1}$$

$$- \frac{n(n-1)}{1.2}\,\frac{(1+\Delta)^{n-2}}{2} + \frac{n(n-1)(n-2)}{1.2.3}\,\frac{(1+\Delta)^{n-3}}{3} - \ldots + (-1)^{n-1}\frac{1}{n}$$

$$= \Gamma(n+1)\left\{\frac{\Delta^{n+1}}{1.2.3\ldots(n+1)} - \frac{\Delta^{n+2}}{2.3.4\ldots(n+2)} + \ldots\right\} \ldots\ldots (A).$$

For example, let each member of the formula

$$D(1+\Delta) - \Delta = \frac{\Delta^2}{1.2} - \frac{\Delta^3}{2.3} + \frac{\Delta^4}{3.4} - \ldots$$

operate on the functions $\frac{\Sigma_x}{x+1}$, $\frac{\Sigma_{x+1}}{x}$, $D\Sigma_x$, and $\frac{2^{-x}}{x}$, then, supposing $x = 0$, we have

$$\frac{\Sigma_2}{1.2.3} + \frac{\Sigma_3}{2.3.4} + \frac{\Sigma_4}{3.4.5} + \ldots = \frac{5}{4} - \frac{\pi^2}{1.2},$$

$$\frac{1}{1.2}\left(\frac{1}{1^2.2}+\frac{1}{2^2.3}\right)+\frac{1}{2.3}\left(\frac{1}{1^2.2}+\frac{1}{2^2.3}+\frac{1}{3^2.4}\right)+\ldots=\tfrac{9}{4}-\frac{\pi^2}{6},$$

$$\frac{\Sigma_2}{1.2^2}+\frac{\Sigma_3}{2.3^2}+\frac{\Sigma_4}{3.4^2}+\ldots=3-2\int_2,$$

(see examples of formula 14 of Art. 1),

$$\frac{1}{1.2}\left(\frac{1}{1.2}+\frac{1}{2.2^2}\right)+\frac{1}{2.3}\left(\frac{1}{1.2}+\frac{1}{2.2^2}+\frac{1}{3.2^3}\right)+\ldots=1-\tfrac{1}{2}\log 2.$$

Operating with (A) upon Σ_x, $\dfrac{1}{(x+1)^v}$, and $\dfrac{\Sigma_{x+1}}{(x+1)^v}$, we find the remarkable summations:

$$(x=m)\qquad \frac{(-1)^n}{\Gamma(m+1)\,\Gamma(n+1)}\left\{\frac{\pi^2}{6}-\Sigma_{m+n}(2)-\Sigma_n\Sigma_{m+n}\right.$$

$$\left.+n\Sigma_{m+n-1}-\frac{n(n-1)}{1.2}\,\frac{\Sigma_{m+n-2}}{2}+\frac{n(n-1)(n-2)}{1.2.3}\,\frac{\Sigma_{m+n-3}}{3}-\ldots+(-1)^{n-1}\frac{\Sigma_m}{n}\right\}$$

$$=\frac{1}{(n+1).1.2.3\ldots(m+n+1)}+\frac{1}{(n+2).2.3.4\ldots(m+n+2)}+\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots(\text{XI}),$$

$$\frac{(-1)^n}{\Gamma(m+1)\,\Gamma(n+1)}\left\{\frac{v}{(m+n+1)^{v+1}}+\frac{\Sigma_n}{(m+n+1)^v}-n\,\frac{1}{(m+n)^v}\right.$$

$$+\frac{n(n-1)}{1.2}\,\frac{1}{2}\,\frac{1}{(m+n-1)^v}-\frac{n(n-1)(n-2)}{1.2.3}\,\frac{1}{3}\,\frac{1}{(m+n-2)^v}$$

$$\left.\ldots+(-)^n\,\frac{1}{n}\,\frac{1}{(m+1)^v}\right\}$$

$$=\frac{\phi_{v-1}(m,n+2)}{1.2.3\ldots(m+n+2)}+\frac{\phi_{v-1}(m,n+3)}{2.3.4\ldots(m+n+3)}+\ldots\ \ldots\ldots(\text{XII}),$$

and $$\frac{(-1)^{n+1}}{\Gamma(m+1)\,\Gamma(n+1)}\left[\frac{1}{(m+n+1)^v}\left\{\frac{\pi^2}{6}-\Sigma_{m+n+1}(2)\right\}\right.$$

$$-v\,\frac{\Sigma_{m+n+1}}{(m+n+1)^{v+1}}-\frac{\Sigma_n\Sigma_{m+n+1}}{(m+n+1)^v}$$

$$+n\,\frac{\Sigma_{m+n}}{(m+n)^v}-\frac{n(n-1)}{1.2}\,\frac{1}{2}\,\frac{\Sigma_{m+n-1}}{(m+n-1)^v}+\ldots+(-)^{n-1}\,\frac{1}{n}\,\frac{\Sigma_{m+1}}{(m+1)^v}$$

$$=\frac{\theta_v(m,n+2)}{1.2.3\ldots(m+n+2)}+\frac{\theta_v(m,n+3)}{2.3.4\ldots(m+n+3)}+\ldots\ \ldots\ldots(\text{XIII}).$$

Let $n=0$ in (XI), then comparing (XI) with (15), we have

$$\frac{\Sigma_1}{1.2.3\ldots(m+2)}+\frac{\Sigma_2}{2.3.4.5\ldots(m+3)}+\ldots$$
$$=\frac{1}{m+1}\left\{\frac{1}{1.1.2.3\ldots(m+1)}+\frac{1}{2.2.3.4\ldots(m+2)}+\ldots\right\}$$
$$=\frac{1}{m+1}\,\frac{1}{\Gamma(m+1)}\left\{\frac{\pi^2}{6}-\Sigma_m(2)\right\}\ldots\ldots\ldots\ldots(\text{XIV}).$$

Minsk.

NOTE.

Mr. Blissard in his paper "On the Sums of Reciprocals" (*Quarterly Mathematical Journal*, Vol. VI., p. 253) has given the result

$$\frac{\Sigma_2(2)}{1.2}+\frac{\Sigma_3(2)}{2.3}+\frac{\Sigma_4(2)}{3.4}+\ldots=\frac{\pi^2}{1.2}+\frac{1}{2},$$

where
$$\Sigma_n(2)=\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\ldots+\frac{1}{n^2}.$$

This summation is inexact. From the formula

$$D\,\frac{\Sigma_{x+1}}{(x+1)^2}=\left\{\frac{\Delta}{1}-\frac{\Delta^2}{2}+\frac{\Delta^3}{3}-\ldots\right\}\frac{\Sigma_{x+1}}{(x+1)^2},$$
$$(\Delta x=1)\left\{\Delta^n\,\frac{\Sigma_{x+1}}{(x+1)^2}\right\}_{x=0}=(-1)^n\,\frac{\Sigma_{n+1}(2)}{n+1},$$

supposing $x=0$, we have

$$\frac{\Sigma_2(2)}{1.2}+\frac{\Sigma_3(2)}{2.3}+\ldots=3-\frac{\pi^2}{6}.*$$

We may obtain the same result, operating with the formula

$$D(1+\Delta)-\Delta=\frac{\Delta^2}{1.2}-\frac{\Delta^3}{2.3}+\ldots$$

upon $\dfrac{\Sigma_x}{x}$ and $\dfrac{\Sigma_x}{x(x+1)}$, and supposing $x=0$.

* The sum of 50 terms in the series in question is more than $\dfrac{\pi^2}{1.2}+\dfrac{1}{2}$.

ON AN ELEMENTARY PROPOSITION IN ATTRACTIONS.

By E. J. ROUTH, M.A., St. Peter's College, Cambridge.

LET M, M' be the masses of two bodies which have the same level surfaces throughout any space. Let V be the potential due to the body M, V' that due to the body M'. Then, throughout the space, when V is constant, V' is constant also, and we must have V' some function of V. Let $V'=f(V)$. Then by differentiating, we easily find

$$\frac{d^2V'}{dx^2}+\frac{d^2V'}{dy^2}+\frac{d^2V'}{dz^2}=\frac{df}{dV}\cdot\left\{\frac{d^2V}{dx^2}+\frac{d^2V}{dy^2}+\frac{d^2V}{dz^2}\right\}$$
$$+\frac{d^2f}{dV^2}\cdot\left\{\left(\frac{dV}{dx}\right)^2+\left(\frac{dV}{dy}\right)^2+\left(\frac{dV}{dz}\right)^2\right\}.$$

Suppose the space considered to be external to both bodies, then by known properties of the potential, this equation reduces to

$$0=\frac{d^2f}{dV^2},$$

unless V be constant throughout the space considered.

This gives $V'=AV+B$, where A and B are two constants.

Suppose the space considered to include the points at infinity, then for such points both V and V' generally vanish. This will be the case when the attracting masses are finite in size and density. Hence $B=0$. Again, V and V' must vanish in the ratio of the attracting masses. Hence $A=\frac{M'}{M}$.

Substituting, we find $\frac{V'}{M'}=\frac{V}{M}$. We have then this theorem.

If two finite bodies of equal mass have the same external level surfaces, then their attractions on all external points are the same in magnitude and direction.

Ex. 1. The external level surfaces of a spherical shell, and of an equal mass collected at its centre, are by symmetry both spheres. Hence the attraction of a spherical shell on any external point, is the same as that of an equal mass collected at its centre.

Ex. 2. Consider the space within an attracting spherical shell. The level surfaces, as before, are the same (viz. spheres) as those of an equal mass at the centre. Hence,

unless the potential V of the shell be constant, we have $V = \frac{A}{r} + B$. But since the attraction decreases, as the distance r of the point from the centre decreases, we must have $A = 0$. Hence, the potential of a spherical shell at all internal points is constant.

Ex. 3. The level surfaces of an infinite plate are planes parallel to the surface, and are not altered by a change of position of the plate parallel to itself. Hence the attraction of an infinite plate on an external point is independent of its distance. The constants may be determined from the condition that at an infinite distance the attractions of two plates separated by a finite interval, tend to equality.

Ex. 4. The level surfaces of an infinite cyclinder are cylinders. Its attraction on any external point is therefore the same as if the whole mass were uniformly distributed along the axis. The constants may be determined as in the last example.

Ex. 5. Let us consider the attraction of an indefinitely thin ellipsoidal shell, bounded by *similar* ellipsoids.

It is proved in Todhunter's *Statics* in an elementary manner, that the attraction on any external point P is along the axis of the enveloping cone whose vertex is P. This direction we know to be a normal to the confocal ellipsoid passing through P. The level surfaces of the ellipsoidal shell are therefore confocal ellipsoids.

It follows that two ellipsoidal shells, as defined above, whose inner surfaces are confocal, have the same level surfaces, and therefore if their masses be equal, exert equal attractions on all external points.

Let us take the two extreme cases of this proposition. Describe an ellipsoidal shell confocal with the given shell, so that the attracted particle P lies on its external surface. It may be proved in an elementary manner,* that the attraction of this shell on P is twice that of an infinite plate, whose thickness is equal to that of the ellipsoidal shell at P. Let abc be the axes of the given attracting shell, $a'b'c'$ the axes of the confocal ellipsoid through P, and let p be the length of the perpendicular from the centre on the tangent plane at P. Then dp is the thickness of the ellipsoidal shell passing through P, and hence $2\pi\mu dp$ is the attraction of

* See the *Solutions of the Senate-House Questions for* 1860, by Watson and Routh, p. 116.

the infinite plate. But the bounding surfaces of the shell are similar, hence $\frac{dp}{p} = \frac{da'}{a'}$; therefore attraction of the ellipsoidal shell $= 4\pi\mu p \frac{da'}{a'}$. But the volume $= 4\pi b'c'da'$, hence

$$\left.\begin{array}{r}\text{attraction of an ellipsoidal shell} \\ \text{on an external point } P\end{array}\right\} = \frac{\text{mass}}{a'b'c'} \cdot p.$$

Ivory's theorem may be deduced from this formula.

Next let us take the other extreme case of the proposition. Let one of the axes of the ellipsoidal shell become zero. Then the shell becomes a disc whose boundary is a focal conic. The density of the disc is not however uniform. Let abc, $a + da$, $b + db$, $c + dc$ be the axes of the two surfaces of the ellipsoidal shell. Then when c vanishes, the mass of any elementary area $dxdy$ of the disc is $2\frac{dz}{dc}dcdxdy$.

Now $z^2 = c^2 - \frac{c^2}{a^2}x^2 - \frac{c^2}{b^2}y^2$, and the surfaces being similar, $\frac{c}{a}$ and $\frac{c}{b}$ are constants, hence

$$\frac{dz}{dc} = \frac{c}{z} = \frac{1}{\sqrt{\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)}}.$$

The density at any point (xy) is therefore *inversely* proportional to $\sqrt{\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)}$.

The attraction of an ellipsoidal shell bounded by similar ellipsoids on any external point is the same as that of an elliptical disc of the same mass bounded by a focal conic, the density at any point P being proportional to $\frac{AB}{\sqrt{(AP.PB)}}$, where AB is the diameter through P.

Ex. 6. If the ellipsoidal shell be bounded by prolate spheroids, the reasoning is a little different. But we may proceed more simply thus. The level surfaces of a straight line are prolate spheroids, having the extremities of the line for foci. These are the same as the level surfaces of the attracting shell. Hence the attraction of a thin shell bounded by similar prolate spheroids on any external point is the same as that of a homogeneous straight line of the same mass joining the foci.

A NEW CONSTRUCTION FOR THE DIFFERENCE OF TWO ANGLES OF A PLANE TRIANGLE.

By J. J. Walker.

A reference in a "Note on the Nine-Point Circle" (*Quarterly Journal*, Vol. VII., p. 302) has reminded me of a property of a plane triangle, suggested by Dr. Salmon's "Geometrical Notes," (*Quarterly Journal*, Vol. IV., p. 153), which has, I believe, not been published hitherto, and may be of interest to those who have a taste for Geometrical constructions. I have not succeeded in obtaining an independent proof, except through the use of Trigonometrical formulæ. Perhaps some reader may supply this deficiency.

I. Suppose ABC to be a plane triangle, O to be the centre of the inscribed circle, and P the point of intersection of perpendiculars AD... let fall from the angles on the opposite sides, and OE a perpendicular from O on BC. Join OD and from E inflect to OD (produced) EF equal to OD; join FP. Then *the angle OFP* (fig. 28) *is equal to half the difference between the angles B and C.*

Call the angle DOE θ, then $DEF = 90° - 2\theta$.

Now $$OE = r = 4R \sin\frac{A}{2} \sin\frac{B}{2} \sin\frac{C}{2};$$

therefore $$DE = 4R \sin\frac{B}{2} \sin\frac{C}{2} \sin\frac{B-C}{2}:*$$

$$PD = 2R \cos B \cos C,$$

and $$\tan\theta = \frac{DE}{OE} = \frac{\sin\frac{B-C}{2}}{\cos\frac{B+C}{2}},$$

whence $$\cot\theta - \tan\theta = \frac{\cos B \cos C}{\sin\frac{A}{2} \sin\frac{B-C}{2}}.$$

* For $DE = OA \sin\frac{B-C}{2} = \frac{r}{\sin\frac{A}{2}} \sin\frac{B-C}{2}$.

But in the triangle PDF,

$$\cot OFP = \frac{DF + DP\cos\theta}{DP\sin\theta} = \frac{DF}{DP\sin\theta} + \cot\theta.$$

Again, from the triangle DEF,

$$\frac{DF}{\sin\theta} = DE\frac{\cos 2\theta}{\sin^2\theta} = DE(\cot^2\theta - 1) = OE(\cot\theta - \tan\theta);$$

therefore
$$\frac{DF}{DP\sin\theta} = \frac{2\sin\frac{B}{2}\sin\frac{C}{2}}{\sin\frac{B-C}{2}};$$

whence finally,
$$\cot OFP = \frac{2\sin\frac{B}{2}\sin\frac{C}{2} + \cos\frac{B+C}{2}}{\sin\frac{B-C}{2}}$$
$$= \cot\frac{B-C}{2}.$$

II. If now FP be produced to meet OA in the point H, the four points $AHDF$ will lie in the circumference of the same circle. Join OP; then, by a known property of an inscribed quadrilateral,

$$OP^2 = OF.OD - AP.PD$$
$$= 2OE^2 - AP.PD$$
$$= 2r^2 - 4R^2\cos A\cos B\cos C,$$

which is the expression given by Dr. Salmon in the Notes above quoted. If the remarkable value of the angle OFP could be established by elementary geometry, the value of OP^2 might therefore be immediately deduced. Similar constructions may of course be given for half the difference of two angles of a triangle by using the radius of the circle escribed to the side lying between those angles; and corresponding values deduced for the square on the line joining P with the centre of that circle.

London, *Feb. 22nd*, 1867.

ON THE EQUILIBRIUM OF A SPHERICAL ENVELOPE.

By J. CLERK MAXWELL, M.A., F.R.S., Trinity College, Cambridge.

I PROPOSE to determine the distribution of stress in an indefinitely thin and inextensible spherical sheet, arising from the action of external forces applied to it at any number of points on its surface.

Notation. Let two systems of lines, cutting each other at right angles, be drawn upon any surface, and let their equations be

$$\phi_1(xyz) = G \text{ and } \phi_2(xyz) = H,$$

where each curve is found by putting G or H equal to a constant, and combining it with the equation of the surface itself, which we may denote by

$$\psi(xyz) = S.$$

Now let G be made constant, and let H vary, and let dS_1 be the element of length of the curve (G = constant) intercepted between the two curves for which H varies by dH, then $\frac{dH}{dS_1}$ will be a function of H and G.

In the same way, making dS_2 an element of the curve (H = constant) we may determine $\frac{dG}{dS_2}$ as a function of H and G.

Now let the element dS_1 experience a stress, consisting of a force X in the direction in which G increases, and a force Y in the direction in which H increases, acting on the positive side of the linear element dS_1, and equal and opposite forces acting on the negative side. These will constitute a longitudinal tension normal to dS_1, which we shall denote by

$$p_{11} = \frac{X}{dS_1},$$

and a shearing force on the element, which we shall call

$$p_{12} = \frac{Y}{dS_1}.$$

In like manner, if the element dS_2 is acted on by forces Y' and X', it will experience on its positive side a tension and a shearing force, the values of which will be

$$p_{22} = \frac{Y'}{dS_2} \text{ and } p_{21} = \frac{X'}{dS}.$$

That the moments of these forces on the element of area dS_1, dS_2 may vanish,

$$p_{12}=p_{21},$$

or the shearing force on dS_1 must be equal to that on dS_2.

When there is no shearing force, then p_{11} and p_{22} are called the *principal* stresses at the point, and if p_{12} vanishes everywhere, the curves, (G = constant) and (H = constant), are called lines of principal stress. In this case the conditions of equilibrium of the element $dS_1 dS_2$ are

$$\frac{dp_{11}}{dG}\frac{dS_1}{dH}+(p_{11}-p_{22})\frac{d^2S_1}{dGdH}=0 \text{ ....(1)},$$

$$\frac{dp_{22}}{dH}\frac{dS_2}{dG}+(p_{22}-p_{11})\frac{d^2S_2}{dGdH}=0 \text{ ....(2)},$$

$$\frac{p_{11}}{r_1}+\frac{p_{22}}{r_2}=N_1-N_2 \text{ .... (3)}.$$

The first and second of these equations are the conditions of equilibrium in the directions of the first and second lines of principal tension respectively.

The third equation is the condition of equilibrium normal to the surface; r_1 and r_2 are the radii of curvature of normal sections touching the first and second lines of principal stress. They are not necessarily the principal radii of curvature. N_1 is the normal pressure of any fluid on the surface from the side on which r_1 and r_2 are reckoned positive, and N_2 is the normal pressure on the other side.

If the systems of curves G and H, instead of being lines of principal stress, had been lines of curvature, we should still have had the same equation (3), but r_1 and r_2 would have been the principal radii of curvature, and p_{11} and p_{22} would have been the tensions in the principal planes of curvature, and not necessarily principal tensions.

In the case of a spherical surface not acted on by any fluid pressure, $r_1=r_2$, and $N_1=N_2=0$, so that the third equation becomes

$$p_{11}+p_{22}=0 \text{ ....(4)},$$

whence we obtain from the first and second equations

$$p_{11}=C_1\left(\frac{dH}{dS_1}\right)^2=-C_2\left(\frac{dG}{dS_2}\right)^2=-p_{22} \text{........(5)},$$

where C_1 is a function of H, and C_2 of G. If then we draw two lines of the system (H = constant) at such a distance that $p_{11}(dS_1)^2=(dh)^2$ at any point where $(dh)^2$ is constant, this equation will continue true through the whole length of these

lines, that is, the principal stresses will be inversely as the square of the distance between the consecutive lines of stress. Since this is true of both sets of lines, we may assume the form of the functions G and H, so that they not only indicate lines of stress, but give the value of the stress at any point by the equations

$$p_{11} = \left(\frac{dH}{dS_1}\right)^2 = \left(\frac{dG}{dS_2}\right)^2 = -p_{22} \ldots\ldots\ldots\ldots (6),$$

where ($H =$ constant) is a line of principal tension, and ($G =$ constant) a line of principal pressure.

If we now draw on the spherical surface lines corresponding to values of G differing by unity, and also lines corresponding to values of H differing by unity, these two systems of lines will intersect everywhere at right angles, and the distance between two consecutive lines of one system will be equal to the distance between two consecutive lines of the other, and the principal stresses will be in the directions of the lines, and inversely as the square of the intervals between them.

Now if two systems of lines can be drawn on a surface so as to fulfil these conditions, we know from the theory of electrical conduction in a sheet of uniform conductivity, that if one set of the curves are taken as equipotential lines, the other set will be lines of flow, and that the two systems of lines will give a solution of some problem relating to the flow of electricity through a conducting sheet. But we know that unless electricity be brought to some point of the sheet, and carried off at another point, there can be no flow of electricity in the sheet. Hence, if such systems of lines exist, there must be some singular points, at which all the lines of flow meet, and at which $\frac{dH}{dS_1}$ is infinite.

If $\frac{dH}{dS_1}$ is nowhere infinite, there can be no systems of lines at all, and if $\frac{dH}{dS_1}$ is infinite, there is an infinite stress at that point, which can only be maintained by the action of an external force applied at that point.

Hence a spherical surface to which no external forces are applied must be free from stress, and it can easily be shewn from this that, when the external forces are given, there can be only one system of stresses in the surface.

This is not true in the case of a plane surface. In a plane surface, equation (3) disappears, and we have only two differential equations connecting the stresses at any point,

which are not sufficient to determine the distribution of stress, unless we have some other condition, such as equations of elasticity, by which the question may be rendered determinate.

The simplest case of a spherical surface acted on by external forces is that in which two equal and opposite forces P are applied at the extremities of a diameter. There will evidently be a tension along the meridian lines, combined with an equal and opposite pressure along the parallels of latitude, and the magnitude p of either of these stresses will be

$$p = \frac{P}{2\pi a \sin^2\theta} \text{ .... } (7),$$

where a is the radius of the sphere, P the force at the poles, and θ the angular distance of a point of the surface from the pole. If ϕ is the longitude, and if r_1 and r_2 are the distances of a point from the two poles respectively, and if we make

$$G = \log_e \frac{r_1}{r_2} \text{ and } H = \phi \text{ .... } (8),$$

then G and H will give the lines of principal stress, and

$$p = \frac{P}{2\pi a}\left(\frac{dG}{dS_2}\right)^2 = \frac{P}{2\pi a}\left(\frac{dH}{dS_1}\right)^2 \text{ .... } (9).$$

To pass from this case to that in which the two forces are applied at any point, I shall make use of the following property of inverse surfaces.

If a surface of any form is in equilibrium under any system of stresses, and if lines of principal stress be drawn on it, then if a second surface be the inverse of the first with respect to a given point, and if lines be drawn on it which are inverse to the lines of principal stress in the first surface, and if along these lines stresses are applied which are to those in the corresponding point of the first surface inversely as the squares of their respective distances from the point of inversion, then every part of the second surface will either be in equilibrium, or will be acted on by a resultant force in the direction of the point of inversion.

For, if we compare corresponding elements of lines of stress in the two surfaces, we shall find that the forces acting on them are in the same plane with the line through the point of inversion, and make equal angles with it. The moment of the force on either element about the point of inversion is therefore as the length of the element, into its distance from the point of inversion, into the intensity of the stress. But the length of the element is as its distance,

and the stress is inversely as the square of the distance, therefore the moments of the stresses about the point of inversion are equal, and in the same plane. If now any portion of the first surface is in equilibrium, it will be in equilibrium as regards moments about the point of inversion. The corresponding portion of the second surface will also be in equilibrium as regards moments about the point of inversion. It is therefore either in equilibrium, or the resultant force acting on it passes through the point of inversion.

Now let the first surface be a sphere; we know that the second surface is also a sphere. In the first surface the condition of equilibrium normal to the surface is $p_{11}+p_{22}=0$. In the second surface the stresses are to those in the first in the inverse ratio of the squares of the distances. Hence in the second surface also, $p_{11}+p_{22}=0$, or there is equilibrium in the direction of the normal. But we have seen that the resultant, if any, is in the radius vector. Therefore, if we except the limiting case in which the radius vector is perpendicular to the normal, the equilibrium is complete in all directions.

We may now, by inverting the spherical surface, pass from the case of a sphere acted on by a tension applied at extremities of a diameter to that in which the forces are applied at the extremities of any chord of the sphere subtending at the centre an angle $=2\alpha$. The lines of tension will be circles passing through the extremities of the chord. Let the angle which one of these circles makes with the great circle through these points be ϕ. The angle ϕ is the same as the corresponding angle in the inverse surface.

The lines of pressure, being orthogonal to these, will be circles whose planes if produced pass through the polar of the chord. Let r_1, r_2 be the distances of any point on the sphere from the extremities of the chord, then $\frac{r_1}{r_2}$ is constant for each of these circles, and has the same value as it has in the inverse surface. Hence, if we make

$$G=\log_a \frac{r_1}{r_2} \text{ and } H=\phi \text{ .... (10)},$$

G and H will give the lines of principal stress, and the absolute value of the stress at any point will be

$$p=\frac{P\sin\alpha}{2\pi a}\left(\frac{dG}{dS_2}\right)^2=\frac{P\sin\alpha}{2\pi a}\left(\frac{dH}{dS_1}\right)^2 \text{ ....(11)}.$$

If we draw tangent planes to the sphere at the extremities

of the chord, and if q_1, q_2 are the perpendiculars from any point on these planes, it is easy to show that at that point

$$p = \frac{Pa \sin^2 \alpha}{2\pi\, q_1 q_2} \quad \text{.....................} (12).$$

If any number of forces, forming a system in equilibrium, be applied at different points of a spherical envelope, we may proceed as follows. First decompose the system of forces into a system of pairs of equal and opposite forces acting along chords of the sphere. To do this, if there are n forces applied at n points, draw a number of chords, which must be at least $3(n-2)$, so as to render all the points rigidly connected. Then determine the tension along each chord due to the external forces. If too many chords have been drawn, some of these tensions will involve unknown quantities. The n forces will now be transformed into as many pairs of equal and opposite forces as chords have been drawn.

Next find the distribution of stress in the spherical surface due to each of these pairs of forces, and combine them at every point by the rules for the composition of stress.* The result will be the actual distribution of stress, and if any unknown forces have been introduced in the process, they will disappear from the result.

The calculation of the resultant stress from the component stresses, when these are given in terms of unsymmetrical spherical coordinates of different systems, would be very difficult; I shall therefore show how to effect the same purpose by a method derived from Mr. Airy's valuable paper, "On the strains in the interior of beams."†

If we place the point of inversion on the surface of the sphere, the inverse surface is a plane, and if p_{xx}, p_{xy}, and p_{yy} represent the components of stress in the plane referred to rectangular axes, we have for equilibrium

$$\frac{dp_{xx}}{dx} + \frac{dp_{xy}}{dy} = 0 \quad \text{..................} \quad (13),$$

$$\frac{dp_{xy}}{dx} + \frac{dp_{yy}}{dy} = 0 \quad \text{..................} \quad (14).$$

These equations are equivalent to the following:

$$p_{xx} = \frac{d^2F}{dy^2}, \quad p_{xy} = -\frac{d^2F}{dx\,dy}, \quad p_{yy} = \frac{d^2F}{dx^2} \quad \text{....} \quad (15),$$

where F is any function whatever of x and y.

* Rankine's *Applied Mechanics*.

† *Philosophical Transactions*, 1863, Part I., p. 49.

The form of the function F cannot, in the case of a plane, be determined from the equations of equilibrium, as strains may exist independently of external forces. To solve the question we require to know not only the original strains, but the law of elasticity of the plane sheet, whether it is uniform, or variable from point to point, and in different directions at the same point. When however we have found two solutions of F corresponding to different cases, we can combine the results by simple addition, as the expressions (equation 15) are linear in form.

In the case of two forces acting on a sphere, let A, B (fig. 29) be the points corresponding to the points of application in the inverse plane; $AP=r_1$, $BP=r_2$, angle $APB=XAT=\phi$. Bisect AB in C, and draw PD perpendicular to AB. Then the line of tension at P is a circle through A and B, for which ϕ is constant, and the line of pressure is an orthogonal circle for which the ratio of r_1 to r_2 is constant, and the angle ϕ and the logarithms of the ratio of r_1 to r_2 differ from the corresponding quantities in the sphere only by constants. We may therefore put

$$p_{11} = \frac{P\sin\alpha}{2\pi a}\left(\frac{dG}{dS_2}\right)^2 = \frac{P\sin\alpha}{2\pi a}\left(\frac{dH}{dS_1}\right)^2 = -p_{22},$$

where the values of G and H are the same as before.

Transforming these principal stresses into their components, we get

$$p_{xx} = \frac{P\sin\alpha}{4\pi a}\left(\left[\frac{dG}{dx}\right]^2 - \left[\frac{dG}{dy}\right]^2\right) \quad\text{.......... (16)},$$

$$p_{xy} = \frac{P\sin\alpha}{4\pi a}\, 2\,\frac{dG}{dx}\,\frac{dG}{dy} \quad\text{.................... (17)},$$

$$p_{yy} = \frac{P\sin\alpha}{4\pi a}\left(\left[\frac{dG}{dx}\right]^2 - \left[\frac{dG}{dx}\right]^2\right) \quad\text{.......... (18)}.$$

From the relations between G and H we have

$$\frac{dG}{dx} = -\frac{dH}{dy} \quad\text{and}\quad \frac{dG}{dy} = -\frac{dH}{dx} \quad\text{......... (19)},$$

whence
$$\frac{d^2G}{dx^2} + \frac{d^2G}{dy^2} = 0 \quad\text{and}\quad \frac{d^2H}{dx^2} + \frac{d^2H}{dy^2} = 0 \quad\text{... (20)}.$$

The values of the component stresses, being expressed as functions of the second degree in G, cannot be compounded

by adding together the corresponding values of G, we must therefore find a value of F, such that

$$\frac{d^2F}{dy^2} = \frac{P\sin\alpha}{2\pi a}\left\{\left(\frac{dG}{dx}\right)^2 - \left(\frac{dG}{dy}\right)^2\right\} = p_{xx} \ldots\ldots (21).$$

We shall then have, by differentiation with respect to x, and integration with respect to y,

$$-\frac{d^2F}{dxdy} = \frac{P\sin\alpha}{2\pi a}\, 2\frac{dG}{dx}\frac{dG}{dy} + f(x) \ldots\ldots\ldots (22),$$

and $$\frac{d^2F}{dx^2} = \frac{P\sin\alpha}{2\pi a}\left\{\left(\frac{dG}{dy}\right)^2 - \left(\frac{dG}{dx}\right)^2\right\} + yf(x) + f_1(x) \ldots (23).$$

Since $\frac{d^2F}{dx^2} + \frac{d^2F}{dy^2} = 0$ in this case, the arbitrary functions must be zero, and we have to find the value of F from that of G by ordinary integration of (21). The result is

$$F = \frac{P\sin\alpha}{2\pi a}\left\{\frac{AD - AC}{AC}\log\frac{AP}{BP} + \frac{PD}{AC}(\text{angle } APB)\right\} \ldots (24),$$

or if the coordinates of A and B are (a_1, b_1) and (a_2, b_2),

$$F = \frac{P\sin\alpha}{2\pi a}\left\{\frac{1}{2}\frac{(2x - a_1 - a_2)(a_2 - a_1) + (2y - b_1 - b_2)(b_2 - b_1)}{(a_2 - a_1)^2 + (b_2 - b_1)^2} \times \log\frac{(x - a_1)^2 + (y - b_1)^2}{(x - a_2)^2 + (y - b_2)^2}\right.$$

$$\left. + \frac{(2y - b_1 - b_2)(a_2 - a_1) - (2x - a_1 - a_2)(b_2 - b_1)}{(a_2 - a_1)^2 + (b_2 - b_1)^2}\left(\tan^{-1}\frac{y - b_2}{x - a_2} - \tan^{-1}\frac{y - b_1}{x - a_1}\right)\right\}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots (25).$$

If we obtain the values of F for all the different pairs of forces acting on the sphere, and add them together, we shall find a new value of F, the second differential coefficients of which with respect to x and y will give a system of components of stress in the plane, which, being transferred to the sphere by the process of inversion, will give the complete solution of the problem in the case of the sphere.

We have now solved the problem in the case of any number of forces applied to points of the spherical surface, and all other cases may be reduced to this, but it is worth while to notice certain special cases.

If two equal and opposite twists be applied at any two points of the sphere, we can determine the distribution of stress. For if we put

$$G = \log \frac{r_1}{r_2} + \phi \text{ and } H = \log \frac{r_1}{r_2} - \phi \qquad (26),$$

the equations of equilibrium will still hold, and the principal stresses at any point will be inclined 45° to those in the case already considered.

If M be the moment of the couple in a plane perpendicular to the chord, the absolute value of the principal stresses at any point is

$$p = \frac{M \sin \alpha}{4\pi a^2} \left(\frac{dG}{dS_2}\right)^2 \qquad (27).$$

In figure 30 are represented the stereographic projections of the principal lines of stress in the cases which we have considered. When a tension is applied along the chord AB, the lines of tension are the circles through AB, and the lines of pressure are the circles orthogonal to them. These circles are so drawn in the figure that the differences of the values of G and H are $\frac{1}{12}\pi$.

The spiral lines which pass through the intersections of these circles are the principal lines of stress in the case of twists applied to the spherical surface at the extremities of the chord AB.

In the case of a sphere acted on by fluid pressure, if the pressure is a function of the distance from a given point, we may take the line joining that point with the centre of the sphere as an axis, and then the lines of equal fluid pressure will be circles, and the fluid pressure N will be a function of θ, the angular distance from the pole. If we suppose the total effect of pressure balanced by a single force at the opposite pole, then we shall have for the equilibrium of the segment whose radius is θ,

$$2 \sin^2 \theta p_{11} = a \int_0^\theta N \sin 2\theta \, d\theta \qquad (28),$$

and

$$p_{22} = Na - p_{11},$$

to determine p_{11} the tension in the meridian, and p_{22} that in the parallels of latitude.

INVESTIGATIONS IN CONNEXION WITH CASEY'S EQUATION.

By Professor Cayley.

In a paper read April 9, 1866, and recently published in the *Proceedings of the Royal Irish Academy*, Mr. Casey has given in a very elegant form the equation of a pair of circles touching each of three given circles, viz. if $U=0$, $V=0$, $W=0$ be the equations of the three given circles respectively, and if considering the common tangents of ($V=0$, $W=0$), of ($W=0$, $U=0$), and of ($U=0$, $V=0$) respectively, these common tangents being such that the centres of similitude through which they respectively pass lie in a line (viz. the tangents are all three direct, or one is direct and the other two are inverse), then if f, g, h are the lengths of the tangents in question, the equation

$$\sqrt{(fU)}+\sqrt{(gV)}+\sqrt{(hW)}=0,$$

belongs to a pair of circles, each of them touching the three given circles. (There are, it is clear, four combinations of tangents, and the theorem gives therefore the equations of four pairs of circles, that is of the eight circles which touch the three given circles).

Generally, if $U=0$, $V=0$, $W=0$ are the equations of any three curves of the same order n, and if f, g, h are arbitrary coefficients, then the equation

$$\sqrt{(fU)}+\sqrt{(gV)}+\sqrt{(hW)}=0,$$

is that of a curve of the order $2n$, touching each of the curves $U=0$, $V=0$, $W=0$, n^2 times, viz. it touches

$$\begin{array}{llll} U=0, & \text{at its } n^2 \text{ intersections with} & gV-hW & =0, \\ V=0 & \quad ,, \qquad\qquad ,, & hW-fU & =0, \\ W=0 & \quad ,, \qquad\qquad ,, & fU-gV & =0. \end{array}$$

If however the curves $U=0$, $V=0$, $W=0$ have a common intersection, then the curve in question has a node at this point, and besides touches each of the three curves in n^2-1 points; and similarly, if the curves $U=0$, $V=0$, $W=0$ have k common intersections, then the curve in question has a node at each of these points, and besides touches each of the three curves in n^2-k points.

In particular, if $U=0$, $V=0$, $W=0$ are conics having two common intersections, then the curve is a quartic having a node at each of the common intersections, and besides touching each of the given conics in two points; whence, if the coefficients f, g, h (that is, their ratios) are so determined that the quartic may have two more nodes, then the quartic, having in all four nodes, will break up into a pair of conics, each passing through the common intersections, and the pair touching each of the given conics in two points; that is, the component conics will each of them touch each of the given conics once. Taking the circular points at infinity for the common intersections, the conics will be circles, and we thus see that Casey's theorem is in effect a determination of the coefficients f, g, h, in such wise that the curve

$$\sqrt{(fU)} + \sqrt{(gV)} + \sqrt{(hW)} = 0,$$

(which when $U=0$, $V=0$, $W=0$ are circles, is by what precedes a bicircular quartic) shall have two more nodes, and so break up into a pair of circles.

The question arises, given $U=0$, $V=0$, $W=0$, curves of the same order n, it is required to determine the ratios $f:g:h$ in such wise that the curve

$$\sqrt{(fU)} + \sqrt{(gV)} + \sqrt{(hW)} = 0,$$

may have two nodes; or we may simply inquire as to the number of the sets of values of $(f:g:h)$, which give a binodal curve, $\sqrt{(fU)} + \sqrt{(gV)} + \sqrt{(hW)} = 0$.

I had heard of Mr. Casey's theorem from Dr. Salmon, and communicated it together with the foregoing considerations to Prof. Cremona, who, in a letter dated Bologna, March 3, 1866, sent me an elegant solution of the question as to the number of the binodal curves. This solution is in effect as follows:

Lemma. Given the curves $U=0$, $V=0$, $W=0$ of the same order n; consider the point (f, g, h), and corresponding thereto the curve $fU+gV+hW=0$. As long as the point (f, g, h) is arbitrary, the curve $fU+gV+hW=0$, will not have any node, and in order that this curve may have a node, it is necessary that the point (f, g, h) shall lie on a certain curve Σ; this being so, the node will lie on a curve J, the Jacobian of the curves U, V, W; and the curves J and Σ will correspond to each other, point to point; viz. taking for (f, g, h) any point whatever on the curve Σ, the curve $fU+gV+hW=0$ will be a curve having a node at some

one point on the curve J; and conversely, in order that the curve $fU+gV+hW=0$ may be a curve having a node at a given point on the curve J, it is necessary that the point (f, g, h) shall be at some one point of the curve Σ. The curve Σ has however nodes and cusps; each node of Σ corresponds to two points of J, viz. the point (f, g, h) being at a node of Σ, the curve $fU+gV+hW=0$, is a binodal curve having a node at each of the corresponding points on J; and each cusp of Σ corresponds to two coincident points of J, viz. the point (f, g, h) being at a cusp of Σ, the curve $fU+gV+hW=0$ is a cuspidal curve having a cusp at the corresponding point of J. The number of the binodal curves $fU+gV+hW=0$ is thus equal to the number of the nodes of Σ, and the number of the cuspidal curves $fU+gV+hW=0$ is equal to the number of the cusps of Σ. The curve Σ is easily shown to be a curve of the order $3(n-1)^2$ and class $3n(n-1)$; and quâ curve which corresponds point to point with J, it is a curve having the same deficiency as J, that is a deficiency $=\frac{1}{2}(3n-4)(3n-5)$; we have thence the Plückerian numbers of the curve Σ, viz.:

Order is	$= 3(n-1)^2,$
Class	$= 3n(n-1),$
Cusps	$=12(n-1)(n-2),$
Nodes	$= \frac{3}{2}(n-1)(n-2)(3n^2-3n-11),$
Inflections	$= 3(n-1)(4n-5),$
Double tangents	$= \frac{3}{2}(n-1)(n-2)(3n^2+3n-8).$

Remarks. The consideration of the foregoing curve Σ is, I believe, first due to Prof. Cremona, it is a curve related to the three distinct curves $U=0$, $V=0$, $W=0$, in the same way precisely as Steiner's curve P_0 is related to the three curves $d_xU=0$, $d_yU=0$, $d_zU=0$. (Steiner, "Allgemeine Eigenschaften der Algebraischen Curven," *Crelle*, t. XLVII. (1854), pp. 1–6; see also Clebsch, "Ueber einige von Steiner behandelte Curven," *Crelle*, t. LXIV. (1865), pp. 288–293), and the Plückerian numbers of P_0 (writing therein $n+1$ for n) are identical with those of Σ. The foregoing expressions $\frac{3}{2}(n-1)(n-2)(3n^2-3n-11)$ and $12(n-1)(n-2)$ for the numbers of the binodal and cuspidal curves $fU+gV+hW=0$, are given in my memoir "On the Theory of Involution," *Cambridge Philosophical Transactions*, t. XI. (1866), pp. 21–38, see p. 32; but the employment of the curve Σ very much simplifies the investigation.

Passing now to the proposed question, we have as before the curves $U=0$, $V=0$, $W=0$, of the same order n; and we may consider the point (f, g, h), and corresponding thereto the curve $\surd(fU)+\surd(gV)+\surd(hW)=0$, say for shortness the curve Ω, which is a curve of the order $2n$, having n^2 contacts with each of the given curves U, V, W. As long as the point (f, g, h) is arbitrary, the curve Ω has not any node; and in order that this curve may have a node, it is necessary that the point (f, g, h) shall lie on a certain curve Δ; this being so, the node will lie on the foregoing curve J, the Jacobian of the given curves U, V, W; and the curves J and Δ will correspond to each other, point to point, viz. taking for (f, g, h) any point whatever on the curve Δ, the curve Ω will have a node at some one point of J; and conversely, in order that the curve Ω may be a curve having a node at a given point of J, it is necessary that the point (f, g, h) shall be at some one point of the curve Δ. The curve Δ has however nodes and cusps; each node of Δ corresponds to two points of J, viz. for (f, g, h) at a node of Δ, the curve Ω is a binodal curve having a node at each of the corresponding points of J; each cusp of Δ corresponds to two coincident points of J, viz. for (f, g, h) at a cusp of Δ, the curve Ω is a cuspidal curve having a cusp at the corresponding point of J. The number of the binodal curves Ω is consequently equal to that of the nodes of Δ, and the number of the cuspidal curves Ω is equal to that of the cusps of Δ; we have consequently to find the Plückerian numbers of the curve Δ; and this Prof. Cremona accomplishes by bringing it into connexion with the foregoing curve Σ, and making the determination depend upon that of the number of the conics which satisfy certain conditions of contact in regard to the curve Σ.

Consider, as corresponding to any given point (f, g, h) whatever, the conic $\dfrac{f}{x}+\dfrac{g}{y}+\dfrac{h}{z}=0$ which passes through three fixed points, the angles of the triangle $x=0$, $y=0$, $z=0$. For points (f, g, h) which lie in an arbitrary line $Af+Bg+Ch=0$, the corresponding conics pass through the fourth fixed point $x:y:z=A:B:C$. Assume for the moment that to the points (f, g, h) which lie on the foregoing curve Δ, correspond conics which touch the foregoing curve Σ. Then 1°. to the points of intersection of the curve Δ with an arbitrary line, correspond the conics which pass through four arbitrary points and touch the curve Σ; or the order of the curve Δ is equal to the number of the conics which can be drawn

through four arbitrary points to touch the curve Σ; viz. if m be the order, n the class of Σ, the number of these conics is $=2m+n$, or substituting for m, n the values $3(n-1)^2$ and $3n(n-1)$ respectively, the number of these conics, that is the order of Δ, is $=3(n-1)(3n-2)$. 2°. To the nodes of Δ correspond the conics which pass through three arbitrary points and have two contacts with Σ, viz. if m be the order, n the class, and κ the number of cusps of Σ, then the number of these conics is $=\frac{1}{2}(2m+n)^2-2m-5n-\frac{3}{2}\kappa$, or substituting for m, n their values as above, and for κ its value $=12(n-1)(n-2)$, the number of these conics, that is, the number of the nodes of Δ, is found to be

$$=\tfrac{3}{2}(n-1)(27n^3-63n^2+22n+16).$$

3°. To the cusps of Δ correspond the conics which pass through three arbitrary points, and have with Σ a contact of the second order; the number of these (m, n, κ as above) is $=3n+\kappa$, or substituting for n and κ their values as above, the number of these conics, that is the number of the cusps of Δ, is $=3(n-1)(7n-8)$. We have thence all the Plückerian numbers of the curve Δ, viz. these are

Order	$= 3(n-1)(3n-2)$,
Class	$= 6(n-1)^2$,
Nodes	$= \frac{3}{2}(n-1)(27n^3-63n^2+22n+16)$,
Cusps	$= 3(n-1)(7n-8)$,
Double tangents	$= \frac{3}{2}(n-1)(12n^3-36n^2+19n+16)$,
Inflexions	$=12(n-1)(n-2)$,

and as a verification it is to be observed, that the deficiency of the curve Δ is equal to that of the curve J, viz. it has the value $\frac{1}{2}(3n-4).(3n-5)$. The foregoing numbers include the result that the number of the binodal curves

$$\sqrt{(fU)}+\sqrt{(gV)}+\sqrt{(hW)}=0,$$

is
$$=\tfrac{3}{2}(n-1)(27n^3-63n^2+22n+16).$$

The proof depended on the assumption, that to the points (f, g, h) which lie on the curve Δ, correspond the conics $\frac{f}{x}+\frac{g}{y}+\frac{h}{z}=0$ which touch the curve Σ; this M. Cremona proves in a very simple manner: the points of J correspond each to each with the points of Σ, or if we please they correspond each to each with the tangents of Σ.

To the $6n(n-1)$ intersections of J with any curve Ω (viz. $\sqrt{(fU)}+\sqrt{(gV)}+\sqrt{(hW)}=0$) correspond the $6n(n-1)$ common tangents of Σ and the conic $\frac{f}{x}+\frac{g}{y}+\frac{h}{z}=0$; if Ω has a node, two of the $6n(n-1)$ intersections coincide, and the corresponding two tangents will also coincide, that is Ω having a node (or the point (f, g, h) being on the curve Δ), the conic touches the curve Σ. But it is not uninteresting to give an independent analytical proof. Write for shortness

$$\begin{aligned} dU &= A\,dx + B\,dy + C\,dz, \\ dV &= A'dx + B'dy + C'dz, \\ dW &= A''dx + B''dy + C''dz, \end{aligned}$$

and let (x, y, z) be the coordinates of a point on J, (X, Y, Z) those of the corresponding point on Σ, (f, g, h) of the corresponding point on Δ. Write also for shortness

$$BC'-B'C,\quad CA'-C'A,\quad AB'-A'B=P:Q:R.$$

Then we have

$$\begin{aligned} AX + BY + CZ &= 0, \\ A'X + B'Y + C'Z &= 0, \\ A''X + B''Y + C''Z &= 0, \end{aligned}$$

$$\begin{array}{llllllll} A\sqrt{\left(\frac{f}{U}\right)} & + & B\sqrt{\left(\frac{g}{V}\right)} & + & C\sqrt{\left(\frac{h}{W}\right)} & = 0, \\ A' \quad ,, & + & B' \quad ,, & + & C' \quad ,, & = 0, \\ A'' \quad ,, & + & B'' \quad ,, & + & C'' \quad ,, & = 0, \end{array}$$

giving $\begin{vmatrix} A, & B, & C \\ A', & B', & C' \\ A'', & B'', & C'' \end{vmatrix}=0$, which is in fact the equation of the curve J; and moreover $X:Y:Z=P:Q:R$, to determine the point (X, Y, Z) on Σ; and

$$\sqrt{\left(\frac{f}{U}\right)}:\sqrt{\left(\frac{g}{V}\right)}:\sqrt{\left(\frac{h}{W}\right)}=P:Q:R,$$

or, what is the same thing, $f:g:h=P^2U:Q^2V:R^2W$, to determine the point (f, g, h) on Δ. Treating now (f, g, h) as constants, and (X, Y, Z) as current coordinates, the conic $\frac{f}{X}+\frac{g}{Y}+\frac{h}{Z}=0$, will touch the curve Σ at the point (P, Q, R), if only the equation of the conic is satisfied by these values

and by the consecutive values $P+dP$, $Q+dQ$, $R+dR$; or what is the same thing, if we have

$$\frac{f}{P}+\frac{g}{Q}+\frac{h}{R}=0,$$

$$\frac{fdP}{P^2}+\frac{gdQ}{Q^2}+\frac{hdR}{R^2}=0,$$

that is

$$\frac{f}{P^2}:\frac{g}{Q^2}:\frac{h}{R^2}=QdR-RdQ:RdP-PdR:PdQ-QdP,$$

or if the functions on the right-hand side are as

$$U:V:W,$$

then these equations give

$$f:g:h=P^2U:Q^2V:R^2W,$$

that is (f, g, h) will be a point on the curve Δ. It is therefore only necessary to show that in virtue of the equation $J=0$ of the curve J, and of the derived equation $dJ=0$, we have

$$QdR-RdQ:RdP-PdR:PdQ-QdP=U:V:W.$$

Take for instance the equation

$$V(QdR-RdQ)-U(RdP-PdR)=0,$$

that is $dR(UP+VQ+WR)-R(UdP+VdQ+WdR)$,

and this, and the other two equations will be satisfied if only $UP+VQ+WR=0$, $UdP+VdQ+WdR=0$; we have, neglecting a numerical factor,

$$U=Ax+A'y+A''z,$$
$$V=Bx+B'y+B''z,$$
$$W=Cx+C'y+C''z,$$

whence, attending to the values of P, Q, R, we have

$$UP+VQ+WR=zJ=0;$$

hence also

$$UdP+VdQ+WdR+(PdU+QdV+RdW)=0,$$

so that $\qquad UdP+VdQ+WdR=0,$

if only $\qquad PdU+QdV+RdW=0,$

and substituting for P, Q, R, dU, dV, dW their values, the left-hand side is $= -Jdz$, which is $= 0$; hence the equations in question are proved, and (f, g, h) is a point on the curve Δ.

It is to be noticed, that the two curves Σ, Δ are geometrically connected through the three arbitrary points as follows: viz. taking as axes the sides of the triangle formed by these three points, then starting from any point (f, g, h) of Δ, we take the inverse point $\left(\frac{1}{f}, \frac{1}{g}, \frac{1}{h}\right)$, the harmonic line thereof $fx + gy + hz = 0$, and finally the inverse conic $\frac{f}{x} + \frac{g}{y} + \frac{h}{z} = 0$, which by what precedes touches Σ in the point corresponding to the assumed point (f, g, h) of Δ; and conversely starting with an assumed point on Σ, we take the conic $\frac{f}{x} + \frac{g}{y} + \frac{h}{z} = 0$ which passes through the angles of the triangle and touches Σ at the assumed point, the inverse line $fx + gy + hz = 0$, the harmonic point $\left(\frac{1}{f}, \frac{1}{g}, \frac{1}{h}\right)$ of the line, and finally the inverse point (f, g, h), which will be on the curve Δ, the point corresponding to the assumed point on the curve Σ.

THEOREM CONCERNING SIX POINTS ON A CIRCLE.

By John Griffiths, M.A.

If five points A, B, C, D, E be taken on the circumference of a circle of radius r, I have shewn (see *The Educational Times*, for March, 1866) that the centres of the five equilateral hyperbolas which pass through them, taken four together, lie on the circumference of another circle of radius $\frac{1}{2}r$. Let this second circle be called, for shortness, the five-centre circle of the points in question; and let a sixth point F be taken on the original circle; then I propose to shew that the five-centre circles of the six groups of points that can be formed from A, B, C, D, E, F, taken five together, will also have their centres on the circumference of a third circle of radius $\frac{1}{4}r$.

Taking any two diameters of the given circle at right angles to each other as axes of reference, we find the coordinates of the centre of the equilateral hyperbola which passes through the four points whose angular ordinates are α, β, γ, δ, to be

$$x = \tfrac{1}{2}r\,(\cos\alpha + \cos\beta + \cos\gamma + \cos\delta),$$
$$y = \tfrac{1}{2}r\,(\sin\alpha + \sin\beta + \sin\gamma + \sin\delta),$$

so that, if the angular ordinates of five given points on the circle be ϕ_1, ϕ_2, ϕ_3, ϕ_4, ϕ_5, the equation of the five-centre circle of these points is

$$\{x - \tfrac{1}{2}r\,(\cos\phi_1 + \cos\phi_2 + \ldots + \cos\phi_5)\}^2$$
$$+ \{y - \tfrac{1}{2}r\,(\sin\phi_1 + \sin\phi_2 + \ldots + \sin\phi_5)\}^2 = \tfrac{1}{4}r^2;$$

and, therefore, the coordinates of its centre are

$$\tfrac{1}{2}r\,(\cos\phi_1 + \cos\phi_2 + \ldots + \cos\phi_5),$$
$$\tfrac{1}{2}r\,(\sin\phi_1 + \sin\phi_2 + \ldots + \sin\phi_5).$$

Hence, given six points ϕ_1, $\phi_2 \ldots \phi_6$ on the circle, the five-centre circles of the six groups which can be formed from them, taken five together, will have their centres at the points

$$\left.\begin{aligned} x &= \tfrac{1}{2}r\,(\cos\phi_1 + \cos\phi_2 + \ldots + \cos\phi_6) - \tfrac{1}{2}r\,\cos\phi_6 \\ y &= \tfrac{1}{2}r\,(\sin\phi_1 + \sin\phi_2 + \ldots + \sin\phi_6) - \tfrac{1}{2}r\,\sin\phi_6 \end{aligned}\right\},$$
$$\left.\begin{aligned} x &= \tfrac{1}{2}r\,(\cos\phi_1 + \cos\phi_2 + \ldots + \cos\phi_6) - \tfrac{1}{2}r\,\cos\phi_1 \\ y &= \tfrac{1}{2}r\,(\sin\phi_1 + \sin\phi_2 + \ldots + \sin\phi_6) - \tfrac{1}{2}r\,\sin\phi_1 \end{aligned}\right\},$$
&c., &c.,

whence it is easily seen that these six centres will lie on the circle whose equation is

$$\{x - \tfrac{1}{2}r\,(\cos\phi_1 + \ldots + \cos\phi_6)\}^2$$
$$+ \{y - \tfrac{1}{2}r\,(\sin\phi_1 + \ldots + \sin\phi_6)\}^2 = \tfrac{1}{4}r^2,$$

which proves the proposition.

Similarly, if the last circle be called the six-centre circle, and if a seventh point be taken on the original circumference, it may be shewn that the seven six-centre circles which can be formed from these points will have their centres on a fourth circle, also of radius $\tfrac{1}{2}r$; and so on.

Jesus College, Oxford,
March 1st, 1867.

ON SOME SPECIAL FORMS OF CONICS.

By C. TAYLOR, M.A., St. John's College, Cambridge.

IN a former article (p. 126) I gave reasons for objecting to the statement that two conjugate points might in certain cases be regarded as a conic. As some writers consider that the various definitions of "conic," which are usually given, are not co-extensive, it may be well to recapitulate certain results which I arrived at.

Results of analytical investigations:

(α) A locus of the second degree cannot degenerate into two conjugate points.

(β) An envelope of the second class cannot degenerate into two conjugate points.

Results of geometrical investigations:

(γ) The locus of a point, the sum or difference of whose distances from two fixed points is constant, cannot degenerate into two conjugate points.

(δ) The envelope of a straight line which cuts four fixed straight lines in a range of constant anharmonic ratio, cannot degenerate into two conjugate points.

In §7, p. 133, I proceeded to show that the above results were "in accordance with the ordinary process of reciprocation;" assuming, that when a hyperbola approaches its asymptotes as a limit, its tangents *all* tend to coincidence with the asymptotes.

Shortly after the publication of the October number of the *Journal*, the following *Erratum* was printed:

The *method* of § 7, p. 133 is incorrect.

The numerical limits of $\tan\theta$, at the origin, are $\frac{b}{a}$ and infinity. The corresponding tangent is indeterminate, but restricted to the *compartment* in which the axis of y lies. The reciprocal is easily proved by elementary geometry to be a terminated straight line.

I propose:

A. To explain more fully the argument which is briefly stated in the erratum.

B. To examine some considerations, by which conclusions to which I objected have been defended.

A.

1. *A pair of straight lines regarded as a limit of a system of hyperbolas, has for tangents all straight lines which pass through the intersection of the two, and are restricted to lie within the same compartment as the conjugate hyperbolas.*

The Cartesian equation

$$x^2 \tan^2 \alpha - y^2 = b^2,$$

represents a hyperbola, whose conjugate axis is equal to $2b$, and whose asymptotes pass through the origin and are inclined at angles α, $\pi - \alpha$, to the axis of x. I propose to examine the limiting form which the curve assumes when b becomes evanescent. The angle α is supposed to remain constant and finite throughout the investigation.

If the tangent at (x', y') be inclined at an angle θ to the axis of x, then

$$\tan\theta = \tan^2\alpha . \frac{x'}{y'}$$

$$= \pm \tan\alpha \sqrt{\left(1 + \frac{b^2}{y'^2}\right)}.$$

When b is evanescent, the expression for $\tan\theta$ becomes numerically equal to $\tan\alpha$, *except when* y' *vanishes.* In this case

$$\tan\theta = \pm \tan\alpha \sqrt{\left(1 + \frac{0}{0}\right)},$$

or $\tan\theta$ lies numerically between $\tan\alpha$ and infinity. It follows that the asymptotes regarded as a limit of the hyperbolas, have for tangents all straight lines which pass through their point of intersection,* and lie within *the same compartment as the conjugate hyperbolas.*

2. *To determine the reciprocal of two straight lines regarded as the limit of a conic.*

There are various methods of reciprocating, all of them leading to results which are essentially the same, though presented under different forms. The method employed in this paragraph is as follows:

Let p (see fig. 31) be the foot of the perpendicular drawn from a fixed point O upon a tangent to that curve, of which

* The equation of the tangent is $xx' \tan^2 a - yy' = b^2$; which reduces to $xx' \tan^2 a - yy' = 0$, when b vanishes. In this case all tangents *pass through the origin.*

the reciprocal is to be found. Divide Op in P, making $Op.OP$ constant. Then will the locus of P be the reciprocal required.

Let the curve to be reciprocated be the straight lines Cf, Cf' (regarded as the limit of a hyperbola), and let Cp be any tangent to the curve. Let F, P, F' be the points which correspond reciprocally to the straight lines Cf, Cp, Cf' respectively. Then P, F, f, p are concyclic, since $OP.Op = OF.Of$; and O, f, p, C are concyclic, since OfC and OpC are right angles.

Therefore
$$\angle OFP = Opf$$
$$= OCf.$$

Hence the angle OFP is constant, and the locus of P is the straight line FF'; the line being limited because the tangent Cp is restricted to one compartment.

If Cp, Cp' be regarded as the limit of the *conjugate* hyperbolas, the tangents will all lie within the compartment pCp', and the reciprocal will evidently be what we may call the complement of FF'; *i.e.* the infinite straight line which can be drawn through F, F', except so much of it as lies between F, F'.

B.

3. M. Chasles has stated, that two straight lines regarded as the limit of a conic, have for tangents "outre ces deux droites mêmes, toutes celles qui passent par leur point d'intersection." By regarding the two lines as at one and the same time, the limit of a hyperbola and its conjugate, we arrive at M. Chasles' conclusion, that any straight line through the intersection of the two lines may be regarded as a tangent. The reciprocal would however be the infinite straight line through F, F', and not simply the two points F, F', as M. Chasles appears to have concluded.

4. Dr. Salmon, after remarking, that if it were not that it is objectionable to multiply names without necessity, it would be convenient to have different names for '*conic* considered as the geometrical interpretation of the general Cartesian, and of the general tangential equation of the second degree;' proceeds as follows:

When the discriminant vanishes, the former is always said to denote two right lines: unless we abandon the whole principle of reciprocity, we must say that when the discriminant vanishes, the latter denotes two points.

But in accordance with the preceding investigations, we may regard two straight lines as a conic, and yet, without abandoning the principle of reciprocity, deny the right of two conjugate points to be so regarded. Dr. Salmon's argument seems indeed to amount to this; viz. that as any one straight line has for its reciprocal a point, so any two straight lines taken at random have for their reciprocals two points. The question—what is the reciprocal of a pair of straight lines regarded as the limit of a conic?—is still left unanswered.

In reciprocating, we pass from a locus to an envelope, or *vicè versa;* and although it may often be convenient in the analytical process of passing from a locus-equation to an envelope-equation, to omit all *direct* consideration of tangents; still the notion of a tangent is necessarily involved. The statement that a conic has degenerated into two straight lines, is insufficient for purposes of reciprocation. We have still to ask: When the curve has degenerated into two straight lines, what has become of its tangents?

(i) The statement that a pair of straight lines has for its reciprocal two conjugate points, involves the assumption that all the tangents to the system are coincident with one or other of the two lines themselves.

(ii) If we assume with M. Chasles, that any straight line whatever through the intersection of the two is a tangent to the system, we must regard the reciprocal as an infinite straight line.

5. I can readily accept Dr. Salmon's conclusion, that when discussing a tangential equation of the form $\lambda\mu = 0$, we must give our name from the envelope. It forms, in fact, the basis of the investigation on p. 128; where the interpretation of this equation is discussed through the medium of Cartesian coordinates. Nor am I conscious of dissenting from the same writer's interpretation of such Cartesian equations as $x^2 + y^2 = 0$. It is only whilst restricting myself altogether to the consideration of real points, that I regard the equation as representing merely a point. The conclusion of §1, p. 126, is not invalidated by the consideration, that a single point might as well be represented by an equation of the fourth or *any* even degree.

6. It will probably be admitted, that results obtained from the consideration of real to the exclusion of imaginary points, may indeed be defective, but are not therefore in-

correct *so far as they go;* and consequently, that if antecedently to the consideration of imaginary points, it can be proved that an equation represents such and such points, we cannot, by merely superadding the conception of imaginary points, take away all or any of the points which were previously represented.

I propose to apply these principles to the discussion of a *particular case* of the equation $\lambda\mu = 0$, which some writers have interpreted as representing nothing but two imaginary points. The equation to which I allude, is the limit (at infinity) of the identical relation in tangential coordinates. The two factors in this case are imaginary.

Mr. Whitworth has shown,* antecedently to the consideration of imaginary points, that the equation spoken of represents a circle of infinite radius and indeterminate centre; which Mr. Whitworth has called "the great circle," and which is identical with what other writers allude to as the "straight line at infinity." This seems to show that it is incorrect to interpret the equation just considered, as representing two imaginary points only; viz. the circular points at infinity. This again seems to indicate the impropriety of interpreting the general equation of the form $\lambda\mu = 0$, as representing nothing but two points.

7. Professor Cayley's "equivalent answer," consists of three statements, of which the following is one:

A conic *qua* curve of the second class can degenerate into a pair of points, but not into a pair of lines.

The first part of this is a statement of what I have all along been controverting. I feel some difficulty in dealing with the second, because I do not know what definition of curve of the second class, I am at liberty to assume. If by a curve of the second class, be meant any curve that can be represented by a tangential quadratic, I would venture to suggest the following difficulties:

It will be granted that a tangential quadratic may be found to represent any given hyperbola with finite axes. Suppose the axes diminished by finite quantities; the resulting curve will have a corresponding equation deducible from the former by certain alterations in the coefficients, and so for a continuous series of variations. Now when we suppose the hyperbola to approach its asymptotes as a limit, we might expect to find a certain limiting relation between the

* *Modern Analytical Geometry*, p. 347.

coefficients corresponding to the limiting form of the curve. In other words, we should anticipate a possibility of proving by means of the equation, that the curve may be made to approach its asymptotes as a limit. Nor am I acquainted with any investigations by which the opposite conclusion has been established.

P.S. The following may be applied to simplify § 2.

Make $OC.Oc$ equal to the constant of reciprocation; and draw cP perpendicular to OC and meeting Op in P. Then P is the reciprocal of Cp, since $OP.Op=OC.Oc$, by similar triangles.

St. John's College,
February, 1867.

ON THE FOCI, AXES, AND ASYMPTOTES OF CONICS, REFERRED TO TRILINEAR COORDINATES.

By Henry M. Jeffery, M.A.

*I. On the Plane Conic.**

1. Implicit equations to determine the foci and asymptotes are given by Mr. Ferrers (*Trilinear Coordinates*, p. 115), and Mr. Hensley has also obtained these equations to the foci and others for the axes, by familiar geometrical theorems (*Quarterly Journal*, Vol. v., pp. 177, 273). Other expressions were found by the writer, whilst investigating the analogues for spherical ellipses, which are rendered explicit by the aid of subsidiary Cartesian coordinates, applied in the case of the poles of the lines of reference. The connection between the two systems, which is thus illustrated, is of some importance in itself.

2. From the greater simplicity of the results, we will first consider the conic inscribed in the triangle of reference, whose equation is

$$\sqrt{(L\alpha)} + \sqrt{(M\beta)} + \sqrt{(N\gamma)} = 0.$$

* This memoir was written before the corresponding discussion of the Spherical Conic, and may be considered introductory to it.

3. To determine the foci of the conic.

Let (f, g, h) be the centre, and ρ_1, ρ_2 the lengths of the semi-axes, major and minor. Then if α_1, α_2 denote the distances of the foci from BC, a side of the triangle of reference, evidently

$$\alpha_1 + \alpha_2 = 2f, \quad \alpha_1\alpha_2 = \rho_2^2.$$

Hence are obtained the usual focal equations

$$\alpha^2 - 2f\alpha = \beta^2 - 2g\beta = \gamma^2 - 2h\gamma = -\rho_2^2.$$

Cor. The equation to the focus of a parabola.

In this case $\frac{\rho_2^2}{f}$ is finite, and $\alpha = f - \left(f - \frac{\rho_2^2}{2f}\right)$ on expansion.

Similarly $$\beta = \frac{\rho_2^2}{2g}, \quad \gamma = \frac{\rho_2^2}{2h}.$$

The equations to the focus are

$$f\alpha = g\beta = h\gamma \text{ or } \frac{L\alpha}{a^2} = \frac{M\beta}{b^2} = \frac{N\gamma}{c^2}$$
$$= -\left(\frac{m^2LMN}{2\Delta abc}\right)^{\frac{1}{3}},$$

as will be shewn in § 5, Cor. 3.

4. Also

$$\alpha = f \pm \sqrt{(f^2 - \rho_2^2)}, \quad \beta = g \pm \sqrt{(g^2 - \rho_2^2)}, \quad \gamma = h \pm \sqrt{(h^2 - \rho_2^2)}.$$

These surds will be quoted as f_1, g_1, h_1, and in like manner f_2, g_2, h_2 will denote $\sqrt{(\rho_1^2 - f^2)}$, $\sqrt{(\rho_1^2 - g^2)}$, $\sqrt{(\rho_1^2 - h^2)}$.

Let f, g, h, the coordinates of the centre, make the angles θ, ϕ, χ with the axis-major; let x_1y_1, x_2y_2, x_3y_3 be the usual Cartesian coordinates (measured from the axes) of the points of contact with the triangle of reference; it follows from familiar theorems, that

$$\cos\theta = \frac{f_1}{\rho_1 e} = \frac{fx_1}{\rho_1^2}, \quad \sin\theta = \frac{f_2}{\rho_1 e} = \frac{fy_1}{\rho_2^2},$$

with similar values for $\cos\phi$, $\sin\phi$, $\cos\chi$, $\sin\chi$.

5. To find the magnitudes of the axes.

The coordinates of the foci must satisfy the fundamental equation

$$a\alpha + b\beta + c\gamma = 2\Delta.$$

But f, g, h must also satisfy this equation.

Hence, $af_1 + bg_1 + ch_1 = 0$. It may also be proved, that $af_2 + bg_2 + ch_2 = 0$. In fact, these results are comprehended in general theorems; for, if the coordinates of any point in a line be inclined to that line at angles θ, ϕ, χ,

$$a \cos\theta + b \cos\phi + c \cos\chi = 0,$$

$$a \sin\theta + b \sin\phi + c \cos\chi = 0,$$

as is evidently true, since a, b, c are proportional to

$$\sin(\phi - \chi), \ \sin(\chi - \theta), \ \sin(\theta - \phi).$$

By expanding the equation thus obtained,

$$a \sqrt{(f^2 - \rho^2)} + b \sqrt{(g^2 - \rho^2)} + c \sqrt{(h^2 - \rho^2)} = 0,$$

we may obtain the form

$$4\Delta^2\rho^4 - \rho^2 abc\,(af^2 \cos A + bg^2 \cos B + ch^2 \cos C)$$
$$+ 4\Delta\,(\Delta - af)\,(\Delta - bg)\,(\Delta - ch) = 0.$$

Cor. 1. The area of the conic

$$= \frac{\pi}{\Delta} \sqrt{\{\Delta\,(\Delta - af)\,(\Delta - bg)\,(\Delta - ch)\}}.$$

Cor. 2. Since $\rho_1^2\rho_2^2$ may be expressed as

$$\Delta^2 \frac{LMN}{abc} \left(\frac{L}{a} + \frac{M}{b} + \frac{N}{c}\right)^{-3},$$

the curve will be a parabola, ellipse, or hyperbola, as

$$\frac{L}{a} + \frac{M}{b} + \frac{N}{c}$$

is zero, positive, or negative.

Cor. 3. To find the focal distance (m) of the parabola.

In the limit $\displaystyle \frac{1}{4m^2} = \frac{\rho_1^2}{\rho_2^4} + \frac{\rho_2^2}{\rho_1^4} = \frac{(\rho_1^2 + \rho_2^2)^3}{\rho_1^4\rho_2^4}.$

Now $$\rho_1^2 + \rho_2^2 = \frac{\Sigma L^2 + 2\Sigma MN \cos A}{4\left(\frac{L}{a} + \frac{M}{b} + \frac{N}{c}\right)^2},$$

$$\rho_1^2\rho_2^2 = \frac{\Delta^2 . LMN}{abc\left(\frac{L}{a} + \frac{M}{b} + \frac{N}{c}\right)^3}.$$

Hence in the limit

$$\frac{1}{4m^2}=\frac{a^2b^2c^2\,\{\Sigma L^2+2\Sigma MN\cos A\}^3}{64\Delta^4L^2M^2N^2}$$

$$=\frac{a^2b^2c^2\left\{L^2\dfrac{bc\cos A}{a^2}+M^2\dfrac{ca\cos B}{b^2}+N^2\dfrac{ab\cos C}{c^2}\right\}}{64\Delta^4L^2M^2N^2}$$

$$=-\frac{LMN}{64abc\Delta^4}\left(\frac{a^3}{L}+\frac{b^3}{M}+\frac{c^3}{N}\right)^3,$$

since $$\frac{L}{a}+\frac{M}{b}+\frac{N}{c}=0.$$

6. To determine the vertices of the conic.

Let p_1, p_2 be the perpendiculars drawn from the extremities of the axis-major on BC, p_3, p_4 those from the ends of the axis-minor, it may be readily proved by Cartesian coordinates, that

$$p_1p_2=\frac{\rho_1^2(f^2-\rho_2^2)}{\rho_1^2-\rho_2^2},\quad p_3p_4=\frac{\rho_2^2(\rho_1^2-f^2)}{\rho_1^2-\rho_2^2}.$$

Hence, as in §3, the coordinates of the extremities are respectively

$$\alpha=f\pm\frac{1}{e}f_1,\quad \beta=g\pm\frac{1}{e}g_1,\quad \gamma=h\pm\frac{1}{e}h_1,$$

$$\alpha=f\pm\frac{b}{ae}f_1,\quad \beta=g\pm\frac{b}{ae}g_1,\quad \gamma=h\pm\frac{b}{ae}h_1.$$

These values might have been at once obtained from the knowledge of the focal coordinates.

Cor. In the parabola, $\dfrac{f(1-e)}{e}$ is constant, when e becomes unity. Hence $\alpha=f-\dfrac{f}{e}\left(1-\dfrac{\rho_2^2}{2f^2}\right)$, on expanding f_1, or $\sqrt{(f^2-\rho_2^2)}$, and

$$\alpha-\frac{\rho_2^2}{2ef}=-\frac{f(1-e)}{e}.$$

The equations to the vertex of a parabola are

$$\frac{1}{f}\left(\alpha-\frac{\rho_2^2}{2f}\right)=\frac{1}{g}\left(\beta-\frac{\rho_2^2}{2g}\right)=\frac{1}{h}\left(\gamma-\frac{\rho_2^2}{2h}\right),$$

or $$\frac{a^2}{L}(\alpha-\alpha_1)=\frac{b^2}{M}(\beta-\beta_1)=\frac{c^2}{N}(\gamma-\gamma_1)=-m\left(\frac{2\Delta.abcm}{LMN}\right)^{\frac{1}{2}},$$

where $\alpha_1, \beta_1, \gamma_1$ are the coordinates of the focus (§3, Cor.) It may also be verified, by bilinear coordinates, that

$$\alpha_1(\alpha_1-\alpha)=\beta_1(\beta_1-\beta)=\gamma_1(\gamma_1-\gamma)=m^2.$$

7. To determine the equations to the axes, major and minor.

By this condition of passing through the foci, the former is

$$\alpha(hg_1-gh_1)+\beta(fh_1-hf_1)+\gamma(gf_1-fg_1)=0.$$

This may be written under the forms, (see §4),

$$\alpha(h\cos\phi-g\cos\chi)+\beta(f\cos\chi-h\cos\theta)+\gamma(g\cos\theta-f\cos\phi)=0,$$

$$\frac{\alpha}{f}(x_2-x_3)+\frac{\beta}{g}(x_3-x_1)+\frac{\gamma}{h}(x_1-x_2)=0.$$

Similarly, the equation to the minor axes may take the forms

$$\alpha(h\sin\phi-g\sin\chi)+\beta(f\sin\chi-h\sin\theta)+\gamma(g\sin\theta-f\sin\phi)=0,$$

$$\frac{\alpha}{f}(y_2-y_3)+\frac{\beta}{g}(y_3-y_1)+\frac{\gamma}{h}(y_1-y_2)=0.$$

Cor. The equation to the major axis of the parabola becomes

$$\alpha\left(\frac{h}{g}-\frac{g}{h}\right)+\beta\left(\frac{f}{h}-\frac{h}{f}\right)+\gamma\left(\frac{g}{f}-\frac{f}{g}\right)=0$$

or $$\alpha\left(\frac{Nb^2}{Mc^2}-\frac{Mc^2}{Nb^2}\right)+\beta\left(\frac{Lc^2}{Na^2}-\frac{Na^2}{Lc^2}\right)+\gamma\left(\frac{Ma^2}{Lb^2}-\frac{Lb^2}{Ma^2}\right)=0.$$

8. It will be shewn in §12, that

$$\frac{x_1-x_2}{\rho_2^2}=\frac{\Delta}{(\Delta-af)(\Delta-bg)}(y_1+y_2),$$

$$\frac{y_1-y_2}{\rho_2^2}=\frac{\Delta}{(\Delta-af)(\Delta-bg)}(x_1+x_2).$$

Hence the equations to the major and minor axes, may be given in these forms:

$$\frac{L\alpha}{af}(y_2+y_3)+\frac{M\beta}{bg}(y_3+y_1)+\frac{N\gamma}{ch}(y_1+y_2)=0.$$

$$\frac{L\alpha}{af}(x_2+x_3)+\frac{M\beta}{bg}(x_3+x_1)+\frac{N\gamma}{ch}(x_1+x_2)=0.$$

9. After multiplying the equations to the axes given in §7, the product may be reduced to the implicit form given by Mr. Hensley,

$$(g^2-h^2)(\alpha-f)^2+(h^2-f^2)(\beta-g)^2+(f^2-g)^2(\gamma-h)^2 \doteq 0.$$

For in this product the coefficient of α^2 is

$$\frac{1}{f^2}(x_2-x_3)(y_2-y_3) \propto \frac{L}{af^2}(x_2^2-x_3^2), \text{ by §8,}$$

$$\propto \frac{L}{af^2}\left(\frac{1}{h^2}-\frac{1}{g^2}\right),$$

since $$\frac{1}{h^2}=\frac{x_3^2}{\rho_1^4}+\frac{y_3^2}{\rho_2^4}.$$

The coefficient of $\dfrac{\beta\gamma}{gh}$ in this product is

$$(y_3-y_1)(x_1-x_2)+(x_3-x_1)(y_1-y_2)$$
$$=(x_3-x_2)(y_3-y_2)-(x_2-x_1)(y_2-y_1)-(x_3-x_1)(y_3-y_1)$$
$$\propto \frac{L}{a}\left(\frac{1}{h^2}-\frac{1}{g^2}\right)-\frac{M}{b}\left(\frac{1}{f^2}-\frac{1}{h^2}\right)-\frac{N}{c}\left(\frac{1}{g^2}-\frac{1}{f^2}\right)$$
$$\propto \frac{c}{h}-\frac{b}{g}-\frac{(ch-bg)}{f^2}.$$

The product now assumes the form

$$\Sigma\alpha^2(g^2-h^2)(\Delta-af)-\Sigma\beta\gamma\{cg(h^2-f^2)+bh(f^2-g^2)\}=0,$$

which may be seen to be identical with the given form.

A simple and direct proof of this implicit form will be given in §27.

10. To find the distances of a point from the axes of the conic.

Take the equation to the major axis

$$\alpha(h\cos\phi-g\cos\chi)+\beta(f\cos\chi-h\cos\theta)+\gamma(g\cos\theta-f\cos\phi)=0.$$

If y denote this distance, the numerator of the fraction is the above function of α, β, γ; the square of the denominator is (Ferrers, p. 21)

$$\Sigma(h\cos\phi-g\cos\chi)^2-2\Sigma\cos A\,(f\cos\chi-h\cos\theta)(g\cos\theta-f\cos\phi).$$

In the expansion, the coefficient of

$$f^2=\cos^2\phi+\cos^2\chi+2\cos\phi\cos\chi\cos A=1-\cos^2(\phi-\chi)=\sin^2 A,$$

since $$\phi-\chi=\pi-A.$$

The coefficient of $2gh$ is

$$\cos^2\theta \cos A - \cos\phi \cos\chi - \cos\theta \cos\phi \cos B - \cos\theta \cos\chi \cos C,$$

which may be reduced to $\sin B \sin C$.

Hence the denominator $= f \sin A + g \sin B + h \sin C = \frac{\Delta}{R}$, if R denote the radius of the circumscribing circle.

Finally, $$\frac{\Delta}{R} y = \alpha (h \cos\phi - g \cos\chi) + \beta (f \cos\chi - h \cos\theta) + \gamma (g \cos\theta - f \cos\phi).$$

Similarly, $$\frac{\Delta}{R} x = \alpha (h \sin\phi - g \sin\chi) + \beta (f \sin\chi - h \sin\theta) + \gamma (g \sin\theta - f \sin\phi).$$

The distance may be thus more simply found. The determinant

$$\begin{vmatrix} \alpha, & \beta, & \gamma \\ f, & g, & h \\ f+f_1, & g+g_1, & h+h_1 \end{vmatrix} = \frac{2\Delta}{R} \times \text{area of } POS.$$

Hence $$\frac{\Delta}{R} y = \begin{vmatrix} \alpha, & \beta, & \gamma \\ f, & g, & h \\ \cos\theta, & \cos\phi, & \cos\chi \end{vmatrix}.$$

Let p, p'; q, q'; r, r'; be the Cartesian coordinates of the vertices of ABC, or, in other words, let p, q, r; p', q', r'; be the tangential coordinates of the minor and major axes

$$h \sin\phi - g \sin\chi = \frac{ap}{2R} = p \sin A.$$

Hence the distances may be thus expressed:

$$2\Delta x = ap\alpha + bq\beta + cr\gamma,$$

$$2\Delta y = ap'\alpha + bq'\beta + cr'\gamma.$$

This is, in fact, the fundamental theorem of transformation, which connects Cartesian with tangential coordinates.

Cor. The equations to the axes may be written

$$\Sigma ap\alpha = 0, \quad \Sigma ap'\alpha = 0.$$

11. The equations to the various forms of the primitive and associated conics, may be obtained by substituting in the equation

$$\frac{x^2}{\rho_1^2} \pm \frac{y^2}{\rho_2^2} = \pm 1,$$

$$\frac{1}{\rho_1^2}\{\Sigma\alpha\,(h \sin\phi - g \sin\chi)\}^2$$

$$\pm \frac{1}{\rho_2^2}\{\Sigma\alpha\,(h \cos\phi - g \cos\chi)\}^2 = \pm (\Sigma\alpha \sin A)^2.$$

12. To find the distances of the points of contact from the axes.

Since the trilinear coordinates of the point of contact with BC, are

$$0,\ \frac{2N\Delta}{Nb+Mc},\ \frac{2M\Delta}{Nb+Mc},\ \text{by §10},$$

$$\frac{Nb+Mc}{2R}\,y_1 = N(f \cos\chi - h \cos\theta) + M(g \cos\theta - f \cos\phi).$$

Or $$Lfy_1 \propto \frac{x_3 - x_1}{Mg} + \frac{x_1 - x_2}{Nh}.$$

Similarly $$Lfx_1 \propto \frac{y_3 - y_1}{Mg} + \frac{y_1 - y_2}{Nh}.$$

13. Hence, it may be shewn that

$$\frac{x_1 - x_2}{\rho_1^2} = \frac{\Delta}{(\Delta - af)(\Delta - bg)}(y_1 + y_2),$$

$$\frac{y_1 - y_2}{\rho_2^2} = \frac{\Delta}{(\Delta - af)(\Delta - bg)}(x_1 + x_2),$$

or $$\frac{g \cos\theta - f \cos\phi}{\rho_2^2} = \frac{\Delta}{(\Delta - af)(\Delta - bg)}(g \sin\theta + f \sin\phi),$$

and $$Mgx_2 + Nhx_3 - Lfx_1 \propto \frac{y_2 - y_3}{Lf},$$

or, if ϕ denote the function $\sqrt{(Lf_1)} + \sqrt{(Mg_1)} + \sqrt{(Nh_1)} = 0$, when cleared of radicals,

$$\frac{y_2 - y_3}{f} \propto \frac{d\phi}{df_1}.\quad \text{So also } \frac{x_2 - x_3}{f} \propto \frac{d\phi}{df_2}.$$

The equations to the major and minor axes may be written

$$\alpha\frac{d\phi}{df_2}+\beta\frac{d\phi}{dg_2}+\gamma\frac{d\phi}{dh_2}=0,$$

$$\alpha\frac{d\phi}{df_1}+\beta\frac{d\phi}{dg_1}+\gamma\frac{d\phi}{dh_1}=0.$$

The points $(f_2g_2h_2)$, $(f_1g_1h_1)$ are the poles of the axes, and are both situated on the line at infinity.

14. In the parabola, $\dfrac{L}{a}y_1=2m\left\{\dfrac{N}{c}\cot C-\dfrac{M}{b}\cot B\right\}$.

Since $\dfrac{Nb+Mc}{2R}y_1\rho_1e=N(fh_1-f_1h)+M(gf_1-g_1f)$,

$\dfrac{Lbcy_1}{2aRm}=N\left(\dfrac{f}{h}-\dfrac{h}{f}\right)+M\left(\dfrac{g}{f}-\dfrac{f}{g}\right)$, since $\dfrac{\rho_2^2}{\rho_1}=2m$, (see §4),

$$=L\frac{c^2-b^2}{a^2}-\frac{Ma}{b}+\frac{Na}{c},$$

$$\frac{Lc\sin By_1}{a}=2m\left\{\frac{N}{c}b\cos C-\frac{M}{b}c\cos B\right\}.$$

Cor. 1. $$\frac{L}{a}y_1+\frac{M}{b}y_2+\frac{N}{c}y_3=0.$$

The fundamental condition is also

$$\frac{L}{a}+\frac{M}{b}+\frac{N}{c}=0.$$

Now by §3, Cor. if (α, β, γ) denotes the focus, these theorems may be written

$$\frac{ay_1}{\alpha}+\frac{by_2}{\beta}+\frac{cy_3}{\gamma}=0,\ \frac{a}{\alpha}+\frac{b}{\beta}+\frac{c}{\gamma}=0.$$

It will be seen that these are instances of the general theorems of §5, for if t_1, t_2, t_3 denote the angles made by the sides of the tangential triangle of reference with the axis, they may be written

$$a\cos t_1+b\cos t_2+c\cos t_3=0,$$

$$a\sin t_1+b\sin t_2+c\sin t_3=0.$$

Cor. 2. Since $\dfrac{L}{a^2\sin t_1}=\dfrac{M}{b^2\sin t_2}=\dfrac{N}{c^2\sin t_3}$,

we can interpret the constants in the tangential form of the equation to the parabola, which may be written

$$\sqrt{(\alpha a^2 \sin t_1)} + \sqrt{(\beta b^2 \sin t_2)} + \sqrt{(\gamma c^2 \sin t_3)} = 0.$$

15. The form of the conic, given in §11, is identical with the usual equation $\sqrt{(L\alpha)} + \sqrt{(M\beta)} + \sqrt{(N\gamma)} = 0$; or, as it may be expressed in terms of the coordinates of its centre,

$$\Sigma a^2\alpha^2 (\Delta - af)^2 - 2\Sigma bc\beta\gamma (\Delta - bg)(\Delta - ch) = 0.$$

Consider the coefficient of α^2; then if the conic be a hyperbola, by §13,

$$\frac{1}{\rho_1^2}(h \sin\phi - g \sin\chi)^2 - \frac{1}{\rho_2^2}(h \cos\phi - g \cos\chi)^2 - \sin^2 A$$

$$= \frac{\Delta}{(\Delta - bg)(\Delta - ch)} \{(h \sin\phi - g \sin\chi)(h \cos\phi + g \cos\chi)$$

$$- (h \cos\phi - g \cos\chi)(h \sin\phi + g \sin\chi)\} - \sin^2 A$$

$$= \frac{2gh\Delta \sin A}{(\Delta - bg)(\Delta - ch)} - \sin^2 A = \frac{\Delta(\Delta - af)}{(\Delta - bg)(\Delta - ch)} \sin^2 A.$$

Next consider the coefficient of $2\beta\gamma$,

$$\frac{1}{\rho_1^2}(f \sin\chi - h \sin\theta)(g \sin\theta - f \sin\phi)$$

$$- \frac{1}{\rho_2^2}(f \cos\chi - h \cos\theta)(g \cos\theta - f \cos\phi) - \sin B \sin C$$

$$= \frac{\Delta}{(\Delta - af)(\Delta - ch)} \{(f \cos\chi + h \cos\theta)(g \sin\theta - f \sin\phi)$$

$$- (f \sin\chi + h \sin\theta)(g \cos\theta - f \cos\phi)\} - \sin B \sin C$$

$$= \frac{\Delta}{(\Delta + af)(\Delta - ch)} \{fh \sin(\theta - \phi)$$

$$- fg \sin(\chi - \theta) - f^2 \sin(\phi - \chi)\} - \sin B \sin C$$

$$= - \frac{\Delta f}{R(\Delta - af)} - \frac{\Delta}{Ra} = - \frac{\Delta \sin B \sin C}{\Delta - af}.$$

16. To determine the asymptotes of the hyperbola.

Since the perpendiculars from a point in an asymptote are proportional to the axes, their equations are

$$\frac{1}{\rho_1} \Sigma\alpha (h \sin\phi - g \sin\chi) = \pm \frac{1}{\rho} \Sigma\alpha (h \cos\phi - g \cos\chi).$$

By § 15, their product is identified with the known implicit form (Ferrers, p. 81),

$$\Sigma\alpha^2 agh(\Delta - af) - 2\Sigma\beta\gamma f(\Delta - bg)(\Delta - ch) = 0.$$

17. To determine the equation to the conjugate hyperbola. Its equation is given in § 11,

$$\frac{1}{\rho_1^2}\{\Sigma\alpha(h\sin\phi - g\sin\chi)\}^2 - \frac{1}{\rho_2^2}\{\Sigma\alpha(h\cos\phi - g\cos\chi)\}^2 + (\alpha\sin A + \beta\sin B + \gamma\sin C)^2 = 0.$$

This is readily reduced to the form

$$\Sigma\alpha^2 agh(\Delta - af) - 2\Sigma\beta\gamma f(\Delta - bg)(\Delta - ch) + \frac{1}{abc}(\Delta - af)(\Delta - bg)(\Delta - ch)(a\alpha + \beta b + \gamma c)^2,$$

or $$\Sigma\alpha^2[L(Lb + Ma)(Lc + Na) + LMNa^2] - 2\Sigma\beta\gamma[MNa(Mc + Nb) - LMNbc] = 0.$$

18. To establish the relation

$$2\Delta(\rho_1^2 + \rho_2^2)\sin 2\theta = c^2(f^2 - h^2) + b^2(g^2 - f^2).$$

By § 4, in the hyperbola

$$2(h^2 - f^2) = (\rho_1^2 + \rho_2^2)(\cos 2\chi - \cos 2\theta) = 2(\rho_1^2 + \rho_2^2)\sin B\sin(\theta + \chi).$$

Hence $c^2(f^2 - h^2) + b^2(g^2 - f^2)$

$$= -(\rho_1^2 + \rho_2^2)\,bc\,\{\sin C\sin(\theta + \chi) + \sin B\sin(\phi + \theta)\}$$
$$= (\rho_1^2 + \rho_2^2)\,bc\sin A\sin 2\theta.$$

19. To find the equation to the coaxial ellipse. By § 11, its equation is

$$\frac{1}{\rho_1^2}\{\Sigma\alpha(h\sin\phi - g\sin\chi)\}^2 + \frac{1}{\rho_2^2}\{\Sigma\alpha(h\cos\phi - g\cos\chi)\}^2 = (\Sigma\alpha\sin A)^2.$$

As before, the coefficient of α^2 may, by the aid of § 13, be reduced to

$$\frac{\Delta}{(\Delta - bg)(\Delta - ch)}(h^2\sin 2\phi - g^2\sin 2\chi) - \sin^2 A$$

$$= -\sin^2 A + \frac{1}{2(\rho_1^2 + \rho_2^2)(\Delta - bg)(\Delta - ch)}\{a^2h^2(g^2 - f^2) + c^2h^2(h^2 - g^2) - b^2g^2(h^2 - g^2) - a^2g^2(f^2 - h^2)\}.$$

Similarly, the coefficient of $2\beta\gamma$ may be reduced to the form

$$-\sin B\sin C+\frac{2\Delta}{a(\Delta-af)}\{g\sin(\theta+\chi)-h\sin(\theta+\phi)\}$$

$$=-\sin B\sin C+\frac{1}{(\rho_1^2+\rho_2^2)(\Delta-af)}\{cg(h^2-f^2)-bh(f^2-g^2)\}.$$

The equation to the coaxial ellipse becomes

$$\begin{aligned}\Sigma\alpha^2(\Delta-af)\{a^2h^2(g^2-f^2)+c^2h^2(h^2-g^2)-b^2g^2(h^2-g^2)-a^2g^2(f^2-h^2)\}\\ +\Sigma 2\beta\gamma(\Delta-bg)(\Delta-ch)\{cg(h^2-f^2)-bh(f^2-g^2)\}\\ -2(\rho_1^2+\rho_2^2)(\Delta-af)(\Delta-bg)(\Delta-ch)(\Sigma\alpha\sin A)^2=0.\end{aligned}$$

Since, by § 5, $\rho_1^2-\rho_2^2$, $-\rho_1^2\rho_2^2$ are known, $\rho_1^2+\rho_2^2$ can be obtained in terms of the given constants.

20. Prob. To find the equation to the ellipse, which shall be inscribed in the triangle of reference, and whose geometrical centre shall be the centre of gravity of the triangle formed by the chords of contact.

With the previous notation, we arrive at the conditions

$$\left.\begin{aligned}x_1+x_2+x_3=0,\quad afx_1+bgx_2+chx_3=0\\ y_1+y_2+y_3=0,\quad afy_1+bgy_2+chy_3=0\end{aligned}\right\},\text{ by § 5.}$$

But, by § 12, the first condition may be expressed

$$(Mg-Nh)x_1+(Nh-Lf)x_2+(Lf-Mg)x_3=0.$$

Hence $$\frac{Mg-Nh}{af}=\frac{Nh-Lf}{bg}=\frac{Lf-Mg}{ch}.$$

This is only possible when $Lf=Mg=Nh$. Hence

$$\frac{L}{a}=\frac{M}{b}=\frac{N}{c},$$

and the equation to the ellipse is

$$\sqrt{(a\alpha)}+\sqrt{(b\beta)}+\sqrt{(c\gamma)}=0,$$

shewing that it touches the sides of the triangle in their middle points.

The centre of gravity of the triangle of reference is also the centre of the ellipse.

21. To extend these processes to the general equation

$$u\alpha^2+v\beta^2+w\gamma^2+2u'\beta\gamma+2v'\gamma\alpha+2w'\alpha\beta=0,$$

the following lemma is necessary.

If p_1, p_2, P, p denote respectively the perpendiculars drawn on any line from the foci, its own pole, and the centre, and $2\rho_2$ the minor axis,

$$p_1 p_2 = \rho_2^2 + pP.$$

The sign of P depends on the side of the polar.

22. To determine the foci of the conic.

Let BC, a side of the triangle of reference, be the polar of the preceding lemma, its pole is given by the equations

$$\frac{\alpha}{U} = \frac{\beta}{W'} = \frac{\gamma}{V'} = \frac{2\Delta}{aU + bW' + cV'},$$

where the capital letters denote minors of the determinant

$$\begin{vmatrix} u, & w', & v' \\ w', & v, & u' \\ v', & u', & w \end{vmatrix}.$$

Hence, if (f, g, h) be the centre of the conic, one of the equations for determining the coordinates of the foci is

$$\alpha^2 - 2f\alpha + \rho_2^2 + \frac{2f\Delta U}{aU + bW' + cV'} = 0,$$

or substituting for f its value, and denoting by D,

$$\Sigma Ua^2 + 2\Sigma U'bc,$$

$$\alpha^2 - 2f\alpha + \rho_2^2 + \frac{4\Delta^2 U}{D} = 0.$$

This, and the other two equations for β, γ, will be recognised as identical with those given by Mr. Ferrers, p. 115.

23. To find the magnitudes of the axes.

For further simplicity denote by f'^2, g'^2, h'^2,

$$\frac{4\Delta^2 U}{D}, \quad \frac{4\Delta^2 V}{D}, \quad \frac{4\Delta^2 W}{D};$$

and by K the discriminant

$$uvw + 2u'v'w' - uu'^2 - vv'^2 - ww'^2.$$

By § 22,

$$\alpha = f \pm \sqrt{(f^2 - f'^2 - \rho_2^2)}, \quad \beta = g \pm \sqrt{(g^2 - g'^2 - \rho_2^2)}, \quad \gamma = h \pm \sqrt{(h^2 - h'^2 - \rho_2^2)}.$$

Substitute these coordinates in the fundamental equation

$$a\alpha + b\beta + c\gamma = 2\Delta,$$

the equation

$$a\sqrt{(f^2-f'^2-\rho_2^2)}+b\sqrt{(g^2-g'^2-\rho_2^2)}+c\sqrt{(h^2-h'^2-\rho_2^2)}=0,$$

gives the magnitude of the axes. Now

$$\begin{aligned}D^2(f'^2-f^2)&=4\Delta^2\{UD-(Ua+W'b+V'c)^2\}\\&=4\Delta^2K(wb^2+vc^2-2u'bc).\end{aligned}$$

After expanding, as in § 5, we may obtain the form of the quadratic

$$\begin{aligned}&16\Delta^2\rho^2-4\rho^2abc\{a\cos A(f^2-f'^2)+b\cos B(g^2-g'^2)+c\cos C(h^2-h'^2)\}\\&\quad+16\Sigma\{\Sigma-a\sqrt{(f^2-f'^2)}\}\{\Sigma-b\sqrt{(g^2-g'^2)}\}\{\Sigma-c\sqrt{(h^2-h'^2)}\}=0,\end{aligned}$$

where $2\Sigma=a\sqrt{(f^2-f'^2)}+b\sqrt{(g^2-g'^2)}+c\sqrt{(h^2-h'^2)}$.

This may be further reduced to the convenient form

$$\begin{aligned}&D^3\rho^4+a^2b^2c^2KD(u+v+w-2u'\cos A-2v'\cos B-2w'\cos C)\rho^2\\&\qquad\qquad+4a^4b^4c^4K^2\Delta^2=0.\end{aligned}$$

Hence follow the familiar theorems for finding the area $(2\pi abcK\Delta D^{-\frac{3}{2}})$, and the condition that the conic be a rectangular hyperbola, Ferrers, p. 83.

In the case of the circle

$$\begin{aligned}\rho^2&=f^2-f'^2=g^2-g'^2=h^2-h'^2 \text{ (Ferrers, p. 85)}\\&=f^2-f\alpha'=g^2-g\beta'=h^2-h\gamma',\end{aligned}$$

as may be recognised geometrically from a diagram.

24. To find the equations to the axes.

Their forms are the same as those given in § 7.

In the most general case, (x_1y_1), (x_2y_2), (x_3y_3) denote the poles of the sides of the triangle of reference; f_1, g_1, h_1 denote the radicals $\sqrt{\left(f^2-\frac{4\Delta^2U}{D}-\rho_2^2\right)}$, ...; f_2, g_2, h_2 denote $\sqrt{\left(\rho_1^2-f^2+\frac{4\Delta^2U}{D}, \dots\right)}$, since the coordinates of the foci are in all cases $f\pm\rho_1e\cos\theta$, $g\pm\rho_1e\cos\phi$, $h\pm\rho_1e\cos\chi$.

It will be shewn that $(f_1g_1h_1)$, $(f_2g_2h_2)$ are the poles of the axes.

25. The following method of determining the positions of the axes corresponds to the process adopted with Cartesian coordinates.

The equation to the conjugate diameter to a system of chords parallel to the line (l, m, n) is

$$\Sigma\alpha \{u (mc - nb) + w' (na - lc) + v' (lb - ma)\} = 0 \ldots (1).$$

That perpendicular to the given system, which passes through the intersection of the perpendiculars drawn from the angles to the opposite sides of reference, is thus defined:

$$\alpha \cos A \left(\frac{m}{b} - \frac{n}{c}\right) + \beta \cos B \left(\frac{n}{c} - \frac{l}{a}\right) + \gamma \cos C \left(\frac{l}{a} - \frac{m}{b}\right) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

If the conjugate diameter be an axis, the equations (1) and (2) only differ by a constant. Equate the coefficients, and multiply by U, W', V' successively; by using undetermined multipliers λ, λ', we have

$$\frac{\lambda}{a}\left(\frac{m}{b} - \frac{n}{c}\right) + \left(\frac{m}{b} - \frac{n}{c}\right) U \cos A + \left(\frac{n}{c} - \frac{l}{a}\right) W' \cos B$$
$$+ \left(\frac{l}{a} - \frac{m}{b}\right) V' \cos C + \lambda' (Ua + W'b + V'c) = 0,$$

with two other equations of like form. Multiply these by a, b, c in succession, and the value of λ' is determined,

$$-2\Delta\lambda' = \left(\frac{m}{b} - \frac{n}{c}\right) f \cos A + \left(\frac{n}{c} - \frac{l}{a}\right) g \cos B + \left(\frac{l}{a} - \frac{m}{b}\right) h \cos C.$$

The eliminant of these three equations is the determinant

$$\begin{vmatrix} \frac{\lambda}{a} + \left(U - \frac{f^2 D}{4\Delta^2}\right)\cos A, & \left(W' - \frac{fgD}{4\Delta^2}\right)\cos B, & \left(V' - \frac{fhD}{4\Delta^2}\right)\cos C \\ \left(W' - \frac{fgD}{4\Delta^2}\right)\cos A, & \frac{\lambda}{b} + \left(V - \frac{g^2 D}{4\Delta^2}\right)\cos B, & \left(U' - \frac{ghD}{4\Delta^2}\right)\cos C \\ \left(V' - \frac{fhD}{4\Delta^2}\right)\cos A, & \left(U' - \frac{ghD}{4\Delta^2}\right)\cos B, & \frac{\lambda}{c} + \left(W - \frac{h^2 D}{4\Delta^2}\right)\cos C \end{vmatrix} = 0.$$

This quadratic may be presented in the form

$$\frac{a}{U' - \frac{ghD}{4\Delta^2} + \mu \cos A} + \frac{b}{V' - \frac{fhD}{4\Delta^2} + \mu \cos B} + \frac{c}{W' - \frac{fgD}{4\Delta^2} + \mu \cos C} = 0.$$

It is readily seen that

$$\left(U - \frac{f^2 D}{4\Delta^2}\right) \frac{D}{K} = wb^2 + vc^2 - 2u'bc,$$

$$\left(U' - \frac{ghD}{4\Delta^2}\right) \frac{D}{K} = -u'a^2 + v'ab + w'ac - ubc.$$

Hence, after reduction, this determinant may be reduced to the form

$$\lambda^2 D + abc\lambda K\Sigma\,(u - 2u' \cos A) + 4\Delta^2 K^2 = 0,$$

which may be identified with the equation in §23, by observing that $\lambda = \dfrac{D\rho^2}{abc}$.

The equations to the axes can be now obtained explicitly.

It will be seen that $\left(\dfrac{m}{b} - \dfrac{n}{c}\right)^2$ is proportional to

$$\left(\frac{\lambda}{b\cos B} + V - \frac{g^2 D}{4\Delta^2}\right)\left(\frac{\lambda}{c\cos C} + W - \frac{h^2 D}{4\Delta^2}\right) - \left(U' - \frac{ghD}{4\Delta^2}\right)^2.$$

Hence $\left(\dfrac{m}{b} - \dfrac{n}{c}\right)^2 bc\cos B\cos C$ varies as

$$\lambda^2 + \frac{\lambda K}{D}\{abc\Sigma\,(u - 2u'\cos A) - a\cos A\,(ub^2 + vc^2 - 2u'bc)\}$$
$$+ \frac{K^2}{D}a^2 bc\cos B\cos C,$$

or, after reducing from the preceding value of λ^2, as

$$-\frac{\lambda K}{D}a\cos A\,(wb^2 + vc^2 - 2u'bc) - \frac{K^2}{D}a^2 bc\cos A.$$

So that $\dfrac{\dfrac{m}{b} - \dfrac{n}{c}}{a\cos A\,\sqrt{\{Kabc + \lambda\,(wb^2 + vc^2 - 2u'bc)\}}} = \ldots = \ldots$

and the equation to an axis takes the form

$$\Sigma\,(\alpha - f)\,a\cos^2 A\,\sqrt{\{Kabc + \lambda\,(wb^2 + vc^2 - 2u'bc)\}} = 0.$$

Cor. If the conic be a circle, this equation is nugatory. In this case (by §23)

$$\lambda\,(wb^2 + vc^2 - 2u'bc) = -\frac{D^3\rho^4}{4\Delta^2 Kabc} = -Kabc.$$

26. To determine the distances from the axes of the poles of the sides of the triangle of reference.

Substitute the coordinates of the pole of BC in the formula of § 10, which is perfectly general, viz.

$$\frac{\alpha}{U} = \frac{\beta}{W'} = \frac{\gamma}{V'},$$

$$\frac{D}{4\Delta R}fy_1 = U(h\cos\phi - g\cos\chi) + W'(f\cos\chi - h\cos\theta)$$
$$+ V'(g\cos\theta - f\cos\phi),$$

with similar expressions for gy_2, hy_3.

Since $\cos\theta$, $\sin\theta$, ... bear the same values as in § 4, the above relation may be presented in the form

$$\frac{D^2\rho_1^{\ 2}}{8KR\Delta^2} fy_1 = (cw' - bv')fx_1 + (cv - bu')gx_2 + (cu' - bw)hx_3.$$

Hence by multiplying the expressions for fy_1, gy_2, hy_3 successively by u, w', v',

$$\begin{aligned}\frac{D^2\rho_1^{\ 2}}{8KR\Delta^2}(ufy_1 + w'gy_2 + v'hy_3) &= gx_2(V'a + U'b + Wc) \\ &\quad - hx_3(W'a + V'b + U'c) \\ &= gh(x_2 - x_3)\frac{D}{2\Delta}.\end{aligned}$$

Hence $$gh(x_2 - x_3) = \frac{D}{8KR\Delta} \cdot \frac{\rho_1\rho_2^{\ 2}}{e} \cdot \frac{d\phi}{df_2}.$$

The equation to the major axis may therefore be written universally

$$\alpha\frac{d\phi}{df_2} + \beta\frac{d\phi}{dg_2} + \gamma\frac{d\phi}{dh_2} = 0.$$

Similarly the equation to the minor axis may be written

$$\alpha\frac{d\phi}{df_1} + \beta\frac{d\phi}{dg_1} + \gamma\frac{d\phi}{dh_1} = 0.$$

The points (f_2, g_2, h_2), (f_1, g_1, h_1) are consequently the poles of the axes.

27. To determine the implicit equation of the axes.

The direct process would find the product of these two explicit equations of the axes, as in § 9; but it is much simpler to obtain the result directly thus:

Let (α, β, γ) be any point in the major axis; $(\alpha', \beta', \gamma')$ either focus; then evidently, since

$$\frac{\alpha - f}{\cos\theta} = \frac{\beta - g}{\cos\phi} = \frac{\gamma - h}{\cos\chi},$$

we have identically

$$\Sigma(\alpha - f)^2\{(\beta' - g)^2 - (\gamma' - h)^2\} = 0.$$

But, by § 22, this may be written

$$\Sigma(\alpha - f)^2\{g^2 - h^2 - g'^2 + h'^2\} = 0,$$

or $$\Sigma(\alpha - f)^2\left\{g^2 - h^2 - \frac{4\Delta^2}{D}(V - W)\right\} = 0,$$

which is Mr. Hensley's general equation.

It may be also presented in the form (§ 23)

$$\Sigma (\alpha - f)^2 \{a^2 (v - w) + b^2 u - c^2 u + 2acv' - 2abw'\} = 0.$$

28. The equation to the axis of the parabola is hence derived by considering that fD, gD, hD are finite,

$$\Sigma \left(\frac{4\Delta^2 U}{D} - 2f\alpha\right)(g^2 - h^2) = 0,$$

or $$\Sigma \{\Delta U - (Ua + W'b + V'c)\, \alpha\} (g^2 - h^2) = 0.$$

This last equation may also be presented in another form, by reducing the general equation to the major axis

$$\alpha (\beta' h - \gamma' g) + \beta (\gamma' f - \alpha' h) + \gamma (\alpha' g - \beta' f) = 0.$$

By § 23 this becomes

$$\Sigma\alpha \left\{\frac{h}{g} (g'^2 + \rho_2^{\,2}) - \frac{g}{h} (h'^2 + \rho_2^{\,2})\right\} = 0.$$

That is, since $\rho_2^{\,2} D = -\dfrac{4K\Delta^2}{\Sigma (u - 2u' \cos A)}$, $f'^2 = \dfrac{4\Delta^2 U}{D}$,

$$\Sigma (u - 2u' \cos A)\, \Sigma\alpha \left(\frac{hV}{g} - \frac{gW}{h}\right) = K \Sigma\alpha \left(\frac{h}{g} - \frac{g}{h}\right).$$

Ex. 1. $u\alpha^2 + v\beta^2 + w\gamma^2 = 0$. For the axis of this parabola,

$$\Sigma a\alpha \left\{\left(1 + \frac{u}{v}\right) b^2 - \left(1 + \frac{u}{w}\right) c^2\right\} = 0.$$

Ex. 2. $a^2\alpha^2 = 4bc\beta\gamma$. The axis is given by the equation

$$a\alpha \frac{b^2 - c^2}{2bc} - \beta (c + b \cos A) + \gamma (b + c \cos A) = 0.$$

From the knowledge of the focal coordinates in § 30, the equation to the parabolic axis may be thus written directly in terms of the coefficients of the quadric

$$\Sigma\alpha \left\{(v'b - w'c) \frac{t_1}{a} + (uc - v'a) \frac{t_2}{b} + (w'a - ub) \frac{t_3}{c}\right\} = 0,$$

where $$t_1 = V + W + 2U' \cos A.$$

29. In the case of the parabola

$$\frac{K}{\Sigma (u - 2u' \cos A)} \left\{\frac{a}{\frac{dD}{da}} + \frac{b}{\frac{dD}{db}} + \frac{c}{\frac{dD}{dc}}\right\} = \frac{aU}{\frac{dD}{da}} + \frac{bV}{\frac{dD}{db}} + \frac{cW}{\frac{dD}{dc}} - 1.$$

By equating the coefficients of the variables in the two equations to the axis of the parabola, given in § 27, we have three equations of the form

$$\frac{\mu a}{\frac{dD}{da}}\Sigma U(g^2-h^2)-\mu(g^2-h^2)=h^2V-g^2W+(g^2-h^2)\frac{K}{\Sigma(u-2u'\cos A)}.$$

By adding these three equations, the indeterminate multiplier (μ) is obtained

$$\frac{1}{2\mu}=\frac{a}{\frac{dD}{da}}+\frac{b}{\frac{dD}{db}}+\frac{c}{\frac{dD}{dc}}.$$

If we substitute this value of μ in any of the three equations and observe that $af+bg+ch=0$, we may obtain the given expression.

30. To find the focus of a parabola.

The coordinates may be thus deduced from those of § 22,

$$\frac{\alpha}{2\Delta}\frac{dD}{da}=U-\frac{K}{\Sigma(u-2u'\cos A)}.$$

Since $$\frac{fD}{\Delta}=\frac{dD}{da},\quad \rho_2^2D=-\frac{4K\Delta^2}{\Sigma(u-2u'\cos A)}.$$

By the aid of § 28, it may be shewn that

$$\left\{U-\frac{K}{\Sigma(u-2u'\cos A)}\right\}\Sigma\left\{\frac{a}{\frac{dD}{da}}\right\}\frac{1}{2abc}\cdot\frac{dD}{db}\cdot\frac{dD}{dc}$$

$$=\frac{U}{a}(V+W+2U'\cos A)+\frac{W'}{b}(W+U+2V'\cos B)$$

$$+\frac{V'}{c}(U+V+2W'\cos C).$$

Hence α varies as the right-hand member of this equation; similar values may be written for β, γ.

31. To find the directrix of a parabola.

Its equation is derived at once from § 30, by considering it as the polar of the focus

$$\Sigma\frac{\alpha}{a}(V+W+2U'\cos A)=0.\quad \text{(Ferrers, p. 88.)}$$

32. To determine the distances, from the axis of a parabola, of the poles of the sides of the triangle of reference.

They may be deduced from the expressions of § 26, by aid of the preceding propositions,

$$y_1 \frac{\Delta}{4ma^2b^2c^2K^2}\left(\frac{dD}{da}\right)^2 \frac{dD}{db}\,\frac{dD}{dc}\left\{\frac{aU}{\frac{dD}{da}}+\frac{bV}{\frac{dD}{db}}+\frac{cW}{\frac{dD}{dc}}-1\right\}$$

$$=\frac{c}{b}(W+U+2V'\cos B)-\frac{b}{c}(U+V+2W'\cos C).$$

The distances of the poles of the two other sides AC, AB, viz. y_2, y_3 have corresponding values.

Cor. $$ay_1\frac{dD}{da}+by_2\frac{dD}{db}+cy_3\frac{dD}{dc}=0.$$

It will be recognised, that, as in the particular case of §14, Cor. 1, this is a form of the general theorem (§ 5)

$$a\sin\theta+b\sin\phi+c\sin\chi=0.$$

The condition that the conic is a parabola, viz.

$$a\frac{dD}{da}+b\frac{dD}{db}+c\frac{dD}{dc}=0,$$

is a case of the other general theorem

$$a\cos\theta+b\cos\phi+c\cos\chi=0,$$

since x_1, x_2, x_3 the Cartesian coordinates of the poles of the sides of ABC, are equal and infinite.

33. To find the latus rectum of a parabola.

Let $4m$ denote its value; then, as in § 5, it may be shewn that

$$\frac{1}{4m^2}=-\frac{a^2b^2c^2}{16\Delta^4K}\{\Sigma(u-2u'\cos A)\}^3.$$

Hence $\dfrac{K}{\Sigma(u-2u'\cos A)}$ may be also written $-\left(2m\dfrac{RK}{\Delta}\right)^{\frac{2}{3}}$.

34. The general equation to the conic may be given in the form

$$\frac{1}{\rho_1^2}(\Sigma l_1\alpha)^2\pm\frac{1}{\rho_2^2}(\Sigma l_2\alpha)^2=(\Sigma\alpha\sin A)^2,$$

where $$l_1=h\sin\phi-g\sin\chi.$$

Or as it may be written (by § 10),

$$\frac{1}{\rho_1^2}(ap\alpha+bq\beta+cr\gamma)^2\pm\frac{1}{\rho_2^2}(ap'\alpha+bq'\beta+cr'\gamma)^2=4\Delta^2.$$

To identify the coefficients in this form with those of the general quadric, *ex. gr.* of α^2. The left-hand side, by § 26,

$$\frac{l_1^2}{\rho_1^2}-\frac{l_2^2}{\rho_2^2}=\frac{D}{4KR\Delta}\,\Sigma l_1\,(u\cos\theta+w'\cos\phi+v'\cos\chi)$$

$$=\frac{D}{4KR\Delta}\left\{-u\begin{vmatrix} f, & g, & h \\ \cos\theta, & \cos\phi, & \cos\chi \\ \sin\theta, & \sin\phi, & \sin\chi\end{vmatrix}+(uf+w'g+v'h)\begin{vmatrix}\cos\phi, & \cos\chi \\ \sin\phi, & \sin\chi\end{vmatrix}\right\}$$

$$=\frac{D}{4KR\Delta}\left\{-u\,(f\sin A+g\sin B+h\sin C)+Ka\sin A\,\frac{2\Delta}{D}\right\}$$

$$=\frac{-uD}{4KR^2}+\sin^2 A.$$

35. To find the equations to the asymptotes.

From the preceding article, the equations are explicitly

$$\frac{1}{\rho_1}\,\Sigma\alpha\,(h\sin\phi-g\sin\chi)=\pm\frac{1}{\rho_2}\,\Sigma\alpha\,(h\cos\phi-g\cos\chi),$$

or if implicitly expressed, by neglecting the term $(\Sigma\alpha\sin A)^2$,

$$(a\alpha+b\beta+c\gamma)^2K-D\phi\,(\alpha,\beta,\gamma)=0:\quad\text{(Ferrers, p. 81).}$$

Cor. The equation to the conjugate hyperbola is

$$2\,(a\alpha+b\beta+c\gamma)^2K-D\phi\,(\alpha,\beta,\gamma)=0,$$

or

$$\phi\,(\alpha,\beta,\gamma)=\frac{8\Delta^2K}{D}.$$

36. To determine the geometrical signification of each coefficient in the general quadric.

The equations to the polar of A and the minor axis being

$$u\alpha+w'\beta+v'\gamma=0,$$

$$\Sigma\alpha\,(h\sin\phi-g\sin\chi)=0,$$

if p_1, q_1, r_1 denote the tangential equations of the polar, ω_1 the inclination of these lines,

$$\frac{ap_1}{u}=\frac{bq_1}{w'}=\frac{cr_1}{v'}=\frac{2\Delta}{\sqrt{\Sigma\,(u^2-2w'v'\cos A)}}=\frac{\mu\cos\omega_1}{ap},$$

if p, q, r be the tangential coordinates of the minor axis.

Hence the general quadric may be written, so as to express the meaning of each coefficient,

$$\Sigma a\alpha p\sec\omega_1\,(p_1a\alpha+q_1b\beta+r_1c\gamma)=0.$$

It may be also shewn that the second form of the quadric, given in § 34, bears this interpretation.

Various geometrical conditions of contact, circumscription, and polarity of the sides and angles of reference, may consequently be expressed in various forms of the quadric, as has been fully explained in the preceding memoir on the Spherical Conic.

37. PROB. To determine the conditions, that the centre of a conic may be the centre of gravity of the triangle formed by the poles of the sides of the triangle of reference.

With the previous notation, we arrive at the conditions

$$\left.\begin{array}{ll} x_1+x_2+x_3=0, & afx_1+bgx_2+chx_3=0 \\ y_1+y_2+y_3=0, & afy_1+bgy_2+chy_3=0 \end{array}\right\} \text{ by § 5.}$$

But by §26, the first condition may be written

$$\Sigma y_1\left(\frac{W}{h^2}-\frac{V}{g^2}+\frac{V'}{fh}-\frac{W'}{fg}\right)=0,$$

since $$fx_1 \propto \frac{U}{f}(y_2-y_3)+\frac{W'}{g}(y_3-y_1)+\frac{V'}{h}(y_1-y_2).$$

Since the sum of the coefficients is zero, this equation can only be consistent with $\Sigma afx_1=0$, if the coefficients separately vanish. Hence, the required conditions are

$$\frac{U}{f^2}-\frac{U'}{gh}=\frac{V}{g^2}-\frac{V'}{fh}=\frac{W}{h^2}-\frac{W'}{fg}.$$

If it be also specified that the centre of gravity of the triangle of reference also coincide with the centre of the conic,

$$Ua^2-U'bc=Vb^2-V'ac=Wc^2-W'ab,$$

since in that case $$af=bg=ch=\frac{2\Delta}{3}.$$

This coincidence of the three centres takes place in the several forms

$$\sqrt{(a\alpha)}+\sqrt{(b\beta)}+\sqrt{(c\gamma)}=0,$$

$$\frac{1}{a\alpha}+\frac{1}{b\beta}+\frac{1}{c\gamma}=0,\quad a^2\alpha^2+2bc\beta\gamma=0.$$

The geometrical reason is easily recognised.

When the triangle of reference is self-conjugate with respect to the conic, and, in fact, coincides with the polar triangle, no case is real.

ON THE EXPANSIBILITY OF THE MULTIPLE INTEGRAL TREATED OF IN VOL. II., PAGE 269.

By Professor SCHLÄFLI.

ASSUMING a system of n orthogonal coordinates $x, y, \ldots z$, measured from an origin O, let the equation

$$x^2+y^2+\ldots+z^2=1$$

(being that of a *n-sphere* having the origin for its centre) be satisfied by n points $1, 2, \ldots n$, that is to say, by n sets of coordinates $(x_1, y_1, \ldots z_1), \ldots, (x_n, y_n, \ldots z_n)$, and suppose them so arranged that their determinant

$$\sqrt{(\Delta)}=\begin{pmatrix} x, y \ldots z \\ 1, 2 \ldots n \end{pmatrix}$$

is positive (I disregard the case where the determinant vanishes). Let the signs of summation Σ, S refer respectively to the coordinates $x, y, \ldots z$, and to the points $1, 2, \ldots n$, and put $\Sigma(x_1-x_2)^2=u_{12}, \ldots$ (so that u_{12}, &c. denote the $\frac{1}{2}n(n-1)$ squares of distances of the n given points on the n-sphere); u_{11} is to be understood to be $=0$. Determine the n linear and homogeneous functions $p_1, p_2, \ldots p_n$ of $x, y, \ldots z$, by the n equations

$$x=Sx_\lambda p_\lambda,\ \ y=Sy_\lambda p_\lambda,\ \ldots,\ \ z=Sz_\lambda p_\lambda\ (\lambda=1, 2, \ldots n),$$

so that $p_1, p_2, \ldots p_n$ may be regarded as oblique coordinates of the point P whose orthogonal ones are $x, y, \ldots z$. Again, let $s=Sp_\lambda$ (so that $s=1$ represents the linear continuum passing through the points $1, 2, \ldots n$), and write

$$U=Su_{\lambda\mu}p_\lambda p_\mu=\tfrac{1}{2}\mathop{S}_{\lambda=1}^{\lambda=n}\mathop{S}_{\mu=1}^{\mu=n}u_{\lambda\mu}p_\lambda p_\mu=R^2\ (R \text{ positive}).$$

Then $r^2=\Sigma x^2=s^2-U$ will be always positive, except at the centre O where it vanishes. Supposing that s, U always denote the same functions of the *independent* variables $x, y, \ldots z$, write $\frac{1}{2}dU=Xdx+\ldots$ (an eventual vanishment of X within the limits of the following integral may be avoided by choosing a system of coordinates where $x_1=x_2=x_3=\ldots=x_n$, that is to say, by taking the axis of x perpendicular to the linear continuum $(123\ldots n)$, whose equation is $s=1$).

Consider now the n-fold integral

$$K=\int dxdy\ldots dz\ (\Sigma x^2<1,\ p_1>0,\ p_2>0,\ \ldots,\ p_n>0),$$

which may be termed a *n-spherical sector*. As is known, this may be expressed in the form

$$K=\frac{1}{n}\int\frac{1}{x}dy\ldots dz\ (\Sigma x^2=1);$$

where I omit the linear limits, which are the same throughout. It is allowable to multiply each element of this integral by

$$\int_0^{\frac{1}{R}} 2R^2 r\,dr = 1,$$

and we thus obtain

$$K = \frac{2}{n}\int \frac{R^2 r}{x}\,dr\,dy \ldots dz\ (0 < rR < 1,\ \Sigma x^2 = 1).$$

Changing then $x, y, \ldots z, R$ into $\frac{x}{r}, \frac{y}{r}, \ldots \frac{z}{r}, \frac{R}{r}$, we have

$$K = \frac{2}{n}\int \frac{R^2}{r^{n-1}x}\,dr\,dy \ldots dz\ (\Sigma x^2 = r^2,\ 0 < R < 1,\ R^2 = s^2 - r^2).$$

Regarding $y, \ldots z$ as constants, we have

$$\frac{rdr}{x} = dx = \frac{RdR}{X},$$

therefore

$$K = \frac{2}{n}\int \frac{R^3}{r^n X}\,dR\,dy \ldots dz\ (0 < R < 1,\ r^2 = \Sigma x^2 = s^2 - R^2),$$

and then replacing $x, y, \ldots z, s$ by $Rx, Ry, \ldots Rz, Rs$, we find

$$K = \frac{2}{n}\int (s^2 - 1)^{-\frac{n}{2}} \frac{RdR}{X}\,dy \ldots dz\ (0 < R < 1,\ \Sigma x^2 = s^2 - 1).$$

We can now integrate in respect to R, and thus obtain

$$K = \frac{1}{n}\int (s^2 - 1)^{-\frac{n}{2}} \frac{1}{X}\,dy \ldots dz\ (\Sigma x^2 = s^2 - 1).$$

But since

$$(s^2 - 1)^{-\frac{n}{2}} = 2\frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} \sum_{\lambda=0}^{\lambda=\infty} \frac{\left(\frac{1}{2}\right)^{n+2\lambda}}{\Gamma(\lambda+1)\,\Gamma\left(\frac{n+1}{2}+\lambda\right)} \int_0^\infty \varepsilon^{-sR} R^{n+2\lambda-1}\,dR,$$

on substituting this expression, replacing $x, y, \ldots z$ by $\frac{x}{R}, \frac{y}{R}, \ldots \frac{z}{R}$, and lastly putting $\frac{RdR}{X} = dx$, we have also

$$K = \frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{n}{2}+1\right)} \sum_{\lambda=0}^{\lambda=\infty} \frac{\left(\frac{1}{2}\right)^{n+2\lambda}}{\Gamma(\lambda+1)\,\Gamma\left(\frac{n+1}{2}+\lambda\right)}$$

$$\times \int \varepsilon^{-s} U^{\lambda}\,dx\,dy \ldots dz\ (U = s^2 - \Sigma x^2),$$

and if $p_1, p_2, \ldots p_n$ be considered as the independent variables,

$$K = \frac{\sqrt{(\Delta)}}{1.2.3\ldots n} \sum_{\lambda=0}^{\lambda=\infty} \frac{(\frac{1}{2})^{2\lambda}}{1.2\ldots\lambda \times \frac{n+1}{2} \cdot \frac{n+3}{2} \cdots \left(\frac{n-1}{2}+\lambda\right)}$$

$$\times \int \varepsilon^{-s} U^{\lambda} dp_1 dp_2 \ldots dp_n \,(p_1 > 0, \ldots).$$

Ultimately U^{λ} can be expanded into a finite series of powers and products of the integration-variables $p_1, p_2, \ldots p_n$, when each term will furnish a combination of the constant elements $u_{12}, \ldots$, multiplied by a product of such simple integrals as $\int_0^\infty \varepsilon^{-p_1} p_1^m dp_1 = 1.2.3\ldots m$. Now as $\frac{\sqrt{(\Delta)}}{1.2\ldots n}$ is nothing else than the integral

$$\int dx dy \ldots dz \,(p_1 + p_2 + \ldots + p_n < 1, p_1 > 0, p_2 > 0, \ldots p_n > 0),$$

that is, the linear polyschemon $(O123\ldots n)$, we may say that the quotient $\frac{\text{sector}}{\text{polyschemon}}$ is expansible into powers and products of the $\frac{n(n-1)}{2}$ squares of distances of their common vertices 1, 2, ... n, which squares are required and sufficient to determine the configuration of the sector.

In order to judge of the convergence of the summatory expression above, we may throw it into the form

$$\frac{1.2\ldots n.K}{\sqrt{(\Delta)}} = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma(\frac{1}{2})\,\Gamma\left(\frac{n}{2}\right)}$$

$$\times \int \varepsilon^{-s} \left(\int_0^\pi \varepsilon^{\sqrt{(U)}.\cos\theta} \sin^{n-1}\theta \, d\theta\right) dp_1 dp_2 \ldots dp_n \,(p_1 > 0, \ldots),$$

and hence infer that its convergence is the same as that of the expression

$$2^{\frac{n}{2}-1} \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma(\frac{1}{2})} \int \varepsilon^{-s+\sqrt{(U)}} U^{-\frac{n}{4}} dp_1 dp_2 \ldots dp_n,$$

in the region where U extends from any very large value up to the value infinity, under the supposition that the integration-variables assume only positive values, and that $\sqrt{(U)}$ is always understood to be positive. I believe therefore that the convergence is secure whenever the quadric function $s^2 - U$ (whose

discriminant is Δ) can be represented by an arithmetical sum of n (and not fewer than n) squares of real linear and homogeneous functions of $p_1, p_2, \ldots p_n$ (such a representation is for instance that by $\Sigma(Sx_\lambda p_\lambda)^2$, if all the linear functions $Sx_\lambda p_\lambda$, $Sy_\lambda p_\lambda$, ... be independent of each other), because then $s^2 - U$ cannot vanish unless each of the independent variables p also vanishes. This condition requires that the discriminant and all its diagonal minors be positive.

Bern, 26 *November*, 1866.

Remark by Prof. Cayley. The interesting theorem established in the above paper will be better understood, by considering that in the case $n=3$, we have in the sphere radius unity a spherical sector $OABC$, standing on the spherical triangle ABC, and that the theorem gives an expression for the spherical sector $OABC \div$ tetrahedron $OABC$, in terms of the rectilineal distances from each other of the points A, B, C which determine the spherical sector; the theorem in its general form being of course the corresponding theorem in n-dimensional geometry.

ON LINEAR DIFFERENTIAL EQUATIONS OF THE THIRD ORDER.

By Chief Justice COCKLE, F.R.S.

(Continued from Vol. VII., p. 326.)

17. I SHALL now examine the conditions under which a linear differential equation of the third order may be deprived of its second and third terms simultaneously, and, in the first instance, I shall assume that, by known processes, the equation has been deprived of its second term; so that, in place of (c), we start from the equation

$$\frac{d^3y}{dx^3} + 3r\frac{dy}{dx} + sy = 0 \;\ldots\ldots\ldots\ldots\ldots\; (B).$$

18. Change the independent variable from x to t, and let the result be

$$\frac{d^3y}{dt^3} + 3Q\frac{d^2y}{dt^2} + 3R\frac{dy}{dt} + Sy = 0 \;\ldots\ldots\ldots\; (C),$$

then (compare Art. 3), we have

$$Q = -\frac{dt}{dx}\frac{d^2x}{dt^2} \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\; (D),$$

$$R = r\left(\frac{dx}{dt}\right)^2 + \left(\frac{dt}{dx}\right)^2\left(\frac{d^2x}{dt^2}\right)^2 - \frac{1}{3}\frac{dt}{dx}\frac{d^3x}{dt^3} \quad \ldots\ldots (E),$$

$$S = s\left(\frac{dx}{dt}\right)^3 \quad \ldots\ldots (F).$$

19. Next, in (C), put

$$y = uY \quad \ldots\ldots (G),$$

then, developing the result and dividing it by u, we obtain an equation which may be written

$$\frac{d^3Y}{dt^3} + 3Q_2\frac{d^2Y}{dt^2} + 3R_2\frac{dY}{dt} + S_2Y = 0 \quad \ldots\ldots (H),$$

and in which

$$Q_2 = Q + \frac{1}{u}\frac{du}{dt} \quad \ldots\ldots (I),$$

$$R_2 = R + \frac{1}{u}\frac{d^2u}{dt^2} + \frac{2Q}{u}\frac{du}{dt} \quad \ldots\ldots (I_2),$$

$$S_2 = S + \frac{1}{u}\frac{d^3u}{dt^3} + \frac{3Q}{u}\frac{d^2u}{dt^2} + \frac{3R}{u}\frac{du}{dt} \quad \ldots\ldots (J).$$

20. In (I) and (I_2) substitute for Q and R their values as given by (D) and (E) respectively, and we have

$$Q_2 = -\frac{dt}{dx}\frac{d^2x}{dt^2} + \frac{1}{u}\frac{du}{dt} \quad \ldots\ldots (K),$$

$$R_2 = r\left(\frac{dx}{dt}\right)^2 + \left(\frac{dt}{dx}\right)^2\left(\frac{d^2x}{dt^2}\right)^2 - \frac{1}{3}\frac{dt}{dx}\frac{d^3x}{dt^3} + \frac{1}{u}\frac{d^2u}{dt^2} - 2\frac{dt}{dx}\frac{d^2x}{dt^2}\frac{1}{u}\frac{du}{dt} \quad \ldots\ldots (L).$$

21. Our object being to annihilate the second and third terms of the transformed equation (H), we make Q_2 and R_2 vanish simultaneously; and

$$Q_2 = 0 \quad \ldots\ldots (M)$$

leads us, through (K), to

$$u = C\frac{dx}{dt} \quad \ldots\ldots (N),$$

where C is an arbitrary constant.

22. In order to satisfy

$$R_2 = 0 \quad \ldots\ldots (O),$$

eliminate u from the dexter of (L). This may be done by means of (N). The result, reduced and equated to zero, is

$$r\left(\frac{dx}{dt}\right)^2 - \left(\frac{dt}{dx}\right)^2\left(\frac{d^2x}{dt^2}\right)^2 + \frac{2}{3}\,\frac{dt}{dx}\,\frac{d^3x}{dt^3} = 0,$$

which is equivalent to

$$2\,\frac{dt}{dx}\,\frac{d^3x}{dt^3} - 3\left(\frac{dt}{dx}\right)^2\left(\frac{d^2x}{dt^2}\right)^2 + 3r\left(\frac{dx}{dt}\right)^2 = 0 \;\ldots.\; (P).$$

23. Now assume

$$\frac{dx}{dt} = p \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\; (Q),$$

whence

$$\frac{d^2x}{dt^2} = \frac{dp}{dt} = \left(\frac{dp}{dx}\right)p, \text{ and } \frac{d^3x}{dt^3} = \left(\frac{d^2p}{dx^2}\right)p^2 + \left(\frac{dp}{dx}\right)^2 p \;\ldots\; (R, S),$$

and eliminate t from (P). That equation becomes

$$2p\,\frac{d^2p}{dx^2} - \left(\frac{dp}{dx}\right)^2 + 3rp^2 = 0 \;\ldots\ldots\ldots\ldots\; (T),$$

and on dividing this result by p^2, we have, after a slight change of form

$$2\,\frac{d}{dx}\left(\frac{1}{p}\,\frac{dp}{dx}\right) + \left(\frac{1}{p}\,\frac{dp}{dx}\right)^2 + 3r = 0 \;\ldots\ldots\; (U).$$

Let

$$\frac{1}{p}\,\frac{dp}{dx} = 2v \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\; (V),$$

then (U) becomes, after division by 4,

$$\frac{dv}{dx} + v^2 + \tfrac{3}{4}r = 0 \;\ldots\ldots\ldots\ldots\ldots\ldots\; (W),$$

or, making

$$e^{\int v dx} = w \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\; (X).$$

$$\frac{d^2w}{dx^2} + \tfrac{3}{4}rw = 0 \;\ldots\ldots\ldots\ldots\ldots\ldots\; (Y),$$

a linear differential equation of the second order.

24. From (X) we deduce

$$v = \frac{1}{w}\,\frac{dw}{dx} \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\; (Z),$$

and, combining this with (V), we find, on integration, &c.

$$p = C_2 w^2 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\; (Z_2),$$

whence, by (Q),

$$\frac{dt}{dx} = \frac{1}{p} = \frac{1}{C_2 w^2} \;\ldots\ldots\ldots\ldots\ldots\ldots\; (Z_3),$$

or $$t = \frac{1}{C_2}\int\frac{dx}{w^2} + C_3 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots (Z_4).$$

We find also, combining (N), (Q) and (Z_2), that

$$u = CC_2w^2 = C_4w^2 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots (Z_5),$$

for the product of the two arbitrary constants is equivalent to one arbitrary constant only.

25. In general (Y) will not be soluble in finite terms, nor in general will (Z_4) enable us to express x as a function of t. In order that we may pass from (B) to

$$\frac{d^3y}{dt^3} + S_2Y = 0 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots (H_2),$$

without employing series, it is sufficient first, that we should be able to integrate the linear differential equation of the second order (Y) finitely, and secondly, either that we should be able, by means of (Z_4), to express x as a function of t, or that the coefficients r and s in (B) should be functions of

$$\int\frac{dx}{w^2},$$

in which case it would be unnecessary to express x as a function of t.*

26. I now proceed to investigations arising out of that new pathway between algebra and the calculus which has been opened out by means of the coresolvents, and in which that theory will, to some extent, be enlarged by the introduction of non-linear differential resolvents. Such non-linear resolvents I have already,† in this part of the world, applied

* In a paper "On Differential Equations and on Co-resolvents," printed at pages 176–196 of the Seventh Volume of the *Transactions of the Royal Society of Victoria* (Melbourne, 1866), I have shown (see Art. 10 of the paper) that the transformation above indicated for equations of the third order is possible for equations of any order.

† In a paper "On the Inverse Problem of Coresolvents," read before the Queensland Philosophical Society on Monday, July 30th, 1866, and printed in the *Queensland Daily Guardian* of Tuesday, August 7th, 1866, I have applied the method of non-linear differential resolvents to the theory of cubic and biquadratic equations. In discussing non-linear resolvents it seems to me absolutely necessary to obtain the expressions of the differential coefficients of the root as rational and entire functions of the root. It was by means of such expressions that I was led to the general theory of coresolvents, and it was by means of such expressions that Mr. Harley was conducted to those results which underlie, either in themselves or as suggestions, so much that has since been effected in that theory. Those expressions are essential parts of the theory, and a con-

to a different problem. I shall, in the present paper, consider them in reference to a linear terordinary, as we may, for the sake of brevity, term a linear differential equation of the third order.

27. Let p, q, and r be the first, second, and third differential coefficients, respectively, of the root, y, of a quintic, whereof the coefficients are functions of the independent variable x. Form the function

$$A\frac{r}{y}+B\frac{pq}{y^2}+C\frac{q}{y}+D\frac{p^3}{y^3}+E\frac{p^2}{y^2}+F\frac{p}{y},$$

then, as we know, this rational function of y, p, q, and r can be expressed as a rational and entire function of y and of the fourth degree in that quantity. By the processes which enter into the theory of coresolvents, put this function under the form

$$Gy^4+Hy^3+Iy^2+Jy+K,$$

wherein the five coefficients G, H, I, J, and K are, each of them, linear and homogeneous functions of the six, here

templation of them leads at once to the conclusion that every rational and entire algebraical equation has a linear differential resolvent of an order less by unity than the degree of the algebraical equation. I mention this because the distinguished Boole (at pp. 738—739 of the *Philosophical Transactions* for 1864) scarcely appears to recognize this conclusion in its full universality. Mr. Harley's formulæ for the resolvents of trinomials with contiguous terms (as we may conveniently designate equations whereof the first, second, and last terms, or the first, penultimate or last but one, and last terms are the only terms which do not vanish) may be combined into one, and, as I have already observed (*Manchester Proceedings*, No. 3, Session 1862-3, p. 17), the differential resolvent of

$$y^n - ny^{n-r} + rx = 0 \dots\dots\dots\dots (a),$$

is, in the cases $r = n - 1$ or $r = 1$,

$$n^{n-1}\left[(n-r)x\frac{d}{dx}\right]^{n-1} y - r^{n-1}\left[nx\frac{d}{dx}-r-1\right]^{n-r-1}$$
$$\times\left[\frac{n}{r}x\frac{d}{dx}-\frac{n+r}{r}\right]^r x^r y = X \dots\dots\dots\dots (b),$$

where $$X = \rho x^r, \ \rho = \frac{1}{n-r}\cdot\frac{d^{n-1}}{dx^{n-1}}x^{n-r} \dots\dots\dots\dots (c).$$

It was the doubts of Mr. Harley (afterwards confirmed, *ibid*) which induced me to confine the theorem to the cases of $r = n - 1$ and $r = 1$; but, although the $(n-1)^{ic}$ resolvent be not, to use the words of Boole (*loc. cit.*), unvarying in its type, still in the two (reciprocal) cases of trinomials with contiguous non-vanishing terms it admits of a unique expression.

supposed indeterminate, quantities A, B, C, D, E, and F. Suppose the four relations

$$G=0,\ H=0,\ I=0,\ J=0 \ldots\ldots\ldots\ldots\ldots(1),$$

to be satisfied. We shall then have the equation

$$A\frac{r}{y}+B\frac{pq}{y^2}+C\frac{q}{y}+D\frac{p^3}{y^3}+E\frac{p^2}{y^2}+F\frac{p}{y}-K=0 \ldots(2),$$

and, the eliminations rendered necessary by the group (1) of equations having been effected, the six quantities A, B, ... F will no longer remain all indeterminate, but any four of them, and K, will be expressible as linear and homogeneous functions of the other two of the originally indeterminate quantities.

28. Transform (2) by means of the exponential substitution or its equivalent

$$\log y=\int u dx \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3),$$

then, since

$$p=\frac{dy}{dx}=yu \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4),$$

$$\left.\begin{aligned} q=\frac{d^2y}{dx^2}&=y\frac{du}{dx}+pu \\ &=y\frac{du}{dx}+yu^2 \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5),$$

$$\left.\begin{aligned} r=\frac{d^3y}{dx^3}&=y\frac{d^2u}{dx^2}+p\frac{du}{dx}+2yu\frac{du}{dx}+pu^2 \\ &=y\frac{d^2u}{dx^2}+3yu\frac{du}{dx}+yu^3 \end{aligned}\right\} \ldots\ldots(6),$$

we perceive that (2) will take the form

$$A\frac{d^2u}{dx^2}+(3A+B)u\frac{du}{dx}+(A+B+D)u^3$$
$$+C\frac{du}{dx}+(C+E)u^2+Fu-K=0 \ldots\ldots\ldots(7).$$

29. Put
$$\left.\begin{aligned} 3A+B\quad &=\alpha \\ A+B+D&=\beta \\ C+E\quad &=\gamma \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots(8),$$

and change the independent variable from x to t. Moreover write

$$\frac{dx}{dt}=w \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(9),$$

then (7) becomes

$$\frac{A}{w^2}\frac{d^2u}{dt^2}+\frac{\alpha}{w}u\frac{du}{dt}+\beta u^3+\left(\frac{C}{w}-\frac{A}{w^3}\frac{dw}{dt}\right)\frac{du}{dt}+\gamma u^2+Fu-K=0 \qquad \text{........ ....(10).}$$

30. Now change the dependent variable from u to U, u and U being connected by the relation

$$u=vU \qquad \text{........................ (11),}$$

then (10) becomes

$$\frac{Av}{w^2}\frac{d^2U}{dt^2}+\frac{\alpha v^2}{w}U\frac{dU}{dt}+\beta v^3U^3$$
$$+\left(\frac{Cv}{w}+\frac{2A}{w^2}\frac{dv}{dt}-\frac{Av}{w^3}\frac{dw}{dt}\right)\frac{dU}{dt}+\left(\gamma v^2+\frac{\alpha v}{w}\cdot\frac{dv}{dt}\right)U^2$$
$$+\left(\frac{A}{w^2}\frac{d^2v}{dt^2}+F_2v\right)U-K=0 \qquad \text{.................... (12),}$$

where $$F_2=\frac{C}{vw}\frac{dv}{dt}-\frac{A}{vw^3}\frac{dv}{dt}\frac{dw}{dt}+F.$$

31. Multiply (12) into the factor

$$\frac{w^2}{Av},$$

and it becomes

$$\frac{d^2U}{dt^2}+\frac{\alpha vw}{A}U\frac{dU}{dt}+\frac{\beta v^2w^2}{A}U^3$$
$$+\left(\frac{Cw}{A}+\frac{2}{v}\frac{dv}{dt}-\frac{1}{w}\frac{dw}{dt}\right)\frac{dU}{dt}+\left(\frac{\gamma vw^2}{A}+\frac{\alpha w}{A}\frac{dv}{dt}\right)U^2$$
$$+\left(\frac{1}{v}\frac{d^2v}{dt^2}+\frac{F_2w^2}{A}\right)U-\frac{Kw^2}{Av}=0 \qquad \text{................(13).}$$

32. Hence if the following conditions are satisfied, viz.

$$\frac{\alpha vw}{A}=3 \qquad \text{......................(14),}$$

$$\frac{\beta v^2w^2}{A}=1 \qquad \text{......................(15),}$$

$$\frac{Cw}{A}+\frac{2}{v}\frac{dv}{dt}-\frac{1}{w}\frac{dw}{dt}=\frac{\gamma vw^2}{A}+\frac{\alpha w}{A}\frac{dv}{dt} \qquad \text{......(16),}$$

it will be possible, as will be seen or shown immediately, to transform (13) into a linear terordinary.

33. Dividing (15) by (14) we have

$$\frac{\beta}{\alpha} vw = \frac{1}{3},$$

therefore
$$vw = \frac{\alpha}{3\beta} \quad \text{.......................(17)},$$

and, eliminating vw from (15) by means of (17), we have further

$$\frac{\beta}{A}\left(\frac{\alpha}{3\beta}\right)^2 = \frac{\alpha^2}{9A\beta} = 1,$$

or
$$9A\beta = \alpha^2 \quad \text{.......................(18)},$$

or, substituting for β and α their respective values as given in the group (8) of Art. 29,

$$9A(A+B+D) = (3A+B)^2 \quad \text{............(19)},$$

whence
$$3A(B+3D) - B^2 = 0 \quad \text{...............(20)}.$$

34. To solve (16) divide it by w, and in the result substitute for vw its value as given by (17), it becomes

$$\frac{C}{A} + \frac{6\beta}{\alpha}\frac{dv}{dt} - \frac{1}{w^2}\frac{dw}{dt} = \frac{\gamma\alpha}{3\beta A} + \frac{\alpha}{A}\frac{dv}{dt} \quad \text{......(21)},$$

or
$$\left(\frac{6\beta}{\alpha} - \frac{\alpha}{A}\right)\frac{dv}{dt} - \frac{1}{w^2}\frac{dw}{dt} = \frac{1}{A}\left(\frac{\gamma\alpha}{3\beta} - C\right) \quad \text{......(22)}.$$

For the sake of convenience, put

$$\frac{3A}{\alpha} = R \quad \text{.......................(23)},$$

then, in virtue of (14),

$$v = \frac{R}{w} \quad \text{.......................(24)},$$

consequently, differentiating,

$$\frac{dv}{dt} = -\frac{R}{w^2}\frac{dw}{dt} + \frac{dR}{dt}\frac{1}{w} \quad \text{.............(25)},$$

and, by means of this last result, eliminating v from (22), we have, after certain easy substitutions,

$$\left(2 - \frac{6\beta}{\alpha}R\right)\frac{1}{w^2}\frac{dw}{dt} + \left(\frac{6\beta}{\alpha} - \frac{3}{R}\right)\frac{dR}{dt}\frac{1}{w} = \frac{\gamma}{\beta}\frac{1}{R} - \frac{C}{A} \quad \text{...(26)}.$$

35. From (9) we have

$$\frac{1}{w} = \frac{dt}{dx} \quad\text{.........................(27)},$$

consequently (26) may be put under the form

$$\left(2 - \frac{6\beta}{\alpha} R\right) \frac{1}{w} \frac{dw}{dx} + \left(\frac{6\beta}{\alpha} - \frac{3}{R}\right) \frac{dR}{dx} = \frac{\gamma}{\beta} \frac{1}{R} - \frac{C}{A} \quad\text{......(28)},$$

and, indeed, it is necessary to put it under such a form, inasmuch as α, β, γ, and R are all given only as functions of x. Multiplying (28) into w and reducing, we have

$$2\left(1 - \frac{3\beta}{\alpha} R\right) \frac{dw}{dx}$$
$$+ \left\{\left(\frac{6\beta}{\alpha} - \frac{3}{R}\right) \frac{dR}{dx} + \frac{C}{A} - \frac{\gamma}{\beta} \frac{1}{R}\right\} w = 0 \quad\text{.........(29)},$$

a linear and homogeneous differential equation in w, whence w may be determined.

36. It is convenient to retain R in the formulæ, but (23) enables us to expel α; and recollecting that

$$\frac{1}{\alpha} = \frac{R}{3A} \quad\text{.........................(30)},$$

we may put (29) under the form

$$2\left(1 - \frac{R^2\beta}{A}\right) \frac{dw}{dx}$$
$$+ \left\{\left(\frac{2R\beta}{A} - \frac{3}{R}\right) \frac{dR}{dx} + \frac{C}{A} - \frac{\gamma}{\beta} \frac{1}{R}\right\} w = 0 \quad\text{......... (31)},$$

or, clearing of fractions,

$$2\,(AR\beta - R^3\beta^2) \frac{dw}{dx}$$
$$+ \left\{(2R^2\beta^2 - 3A\beta) \frac{dR}{dx} + CR\beta - A\gamma\right\} w = 0 \quad\text{.........(32)},$$

and if

$$A\beta - (R\beta)^2 = S \quad\text{.................. (33)},$$

we may put (32) under the form

$$2RS \frac{dw}{dx} + \left\{(R^2\beta^2 - 3S) \frac{dR}{dx} + CR\beta - A\gamma\right\} w = 0 \quad\text{...(34)}.$$

37. The solution of this linear differential equation will enable us to express w as a function of x. Then from (9) we obtain

$$\frac{dt}{dx}=\frac{1}{w} \quad\text{.......................(35)},$$

whence

$$t=\int\frac{dx}{w}\ [=\phi(x),\ \text{say}] \quad\text{............... (36)},$$

and if from $t=\phi(x)$ we can obtain $x=\phi^{-1}(t)$, where $\phi^{-1}(t)$ is assignable without series, or if all the functions of x involved in our formulæ are also functions of $\phi(x)$, or, what is the same thing, of $\int\frac{dx}{w}$, so that in fact it is unnecessary to express x as a function of t; then in such cases we shall be able to satisfy (34) and consequently (16) finitely, or by recognized assignable functions.

38. Indeed, the relation (18) does not involve either v or w, and will have to be satisfied by means of the indeterminate quantities. But the equation (18), though homogeneous in those quantities, may still be satisfied, for the solution of the four equations which constitute the group (1) leaves us one disposable ratio to be used in the solution of (18). It will be remembered that the original indeterminates were six in number. Then, (18) being solved, and v eliminated from (16) by the processes above indicated, and w determined, v may be obtained from (14) or (15).

39. We are thus conducted to a class of cases in which the solution of a quintic may be made to depend upon that of a linear terordinary. I may add that whenever the coefficients of any terordinary of the form of (2) satisfy the relation (20), then (apart from any derivation of it from a quintic) such terordinary may be transformed into a linear terordinary. So far as the assignability of $\phi^{-1}(x)$ is concerned, the same points as before arise.

40. It remains to exhibit the actual form of the linear terordinary. To this end let

$$\log V=\int Udx,\ \text{ or } V=e^{\int Udx} \quad\text{............ (37)},$$

and multiply (13) into V. Then bearing in mind the formulæ in (3), (4), (5), and (6), and the relations indicated by

(14), (15), and (16), we see that there will arise a resulting equation in V of the following form:

$$\frac{d^3V}{dt^3}+\left(\frac{Cw}{A}+\frac{2}{v}\cdot\frac{dv}{dt}-\frac{1}{w}\cdot\frac{dw}{dt}\right)\frac{d^2V}{dt^2}$$

$$+\left(\frac{1}{v}\cdot\frac{d^2v}{dt^2}+\frac{F_2w^2}{A}\right)\frac{dV}{dt}-\frac{Kw^2}{Av}V=0\ldots\ldots(38),$$

a linear terordinary.

"Oakwal," near Brisbane,
Queensland, Australia,
January 15, 1867.

(To be continued).

NOTICE.

THE following circular has been issued, announcing a change of the Editorship and mode of Publication of the

"ANNALI DI MATEMATICA PURA ED APPLICATA."

Milano, 10 febbrajo, 1867.

CHIARISSIMO SIGNORE,

Dietro accordi stabiliti fra i sottoscritti ed i professori Tortolini, Betti e Genocchi, gli *Annali di matematica pura ed applicata*, sinora pubblicati in Roma dal signor Tortolini, lo saranno d'ora in poi in Milano, dove usciranno per fascicoli liberi (senza vincolo-di tempo, a somiglianza del Giornale Crelle-Borchardt), ciascuno di 10 o 12 fogli in 4.°* Editori sono i sottoscritti, ma continua la cooperazione dei tre chiarissimi professori nominati, coll' aggiunta di altri geometri italiani e stranieri.

I sottoscritti, aprendo le pagine degli *Annali* a tutti i cultori delle scienze matematiche, confidano che V. S.

* Quattro fascicoli formeranno un volume, che costerà lire 16, senz'aggiunta di spese postali e senza distinzione di paese. Questa somma dev'essere mandata per vaglia postale, o fatta pagare col mezzo di un corrispondente, al *tipografo sig. Francesco Zanetti, via del Senato* 26 *Milano*, al quale si ricolgeranno le domande di associazione.

Se V. S. desidera che il suo nome o quello di alcuno de' suoi amici sia inscritto fra gli associati, è pregata di darne avviso affinchè si possa stabilire il numero delle copie da stamparsi.

Chiar, vorrà contribuirvi lavori proprî, e le saranno molto grati se procaccerà a questo periodico anche la collaborazione de' suoi amici. Pregano poi V. S. di comunicare loro i nomi delle persone che accettano l'invito, affinchè si possa al più presto farli noti al pubblico.

I manoscritti si mandino affrancati ad uno degli scriventi o alla *Direzione degli Annali di matematica*, il cui ricapito comune è *presso il R. Istituto Tecnico Superiora di Milano.* Le memorie non saranno pubblicate che in lingua italiana, latina o francese; ma se V. S. preferisse servirsi della tedesca o dell'inglese, qui si provvedera alla traduzione. Gli autori riceveranno senza spese 25 copie separate delle loro memorie inserite negli *Annali.*

Gli editori dei periodici matematica, che hanno fatto sinora il cambio cogli *Annali*, sono pregati di mandare quind'innanzi i loro fascicoli, di mano in mano che vengono alla luce, ad uno de'sottoscritti o alla *Direzione degli Annali di matematica.*

Dalla cortesia dei direttori di giornali, riviste, ecc. i quali riceveranno questa lettera circolare, si attende che ad essa sia data la maggiore possibile pubblicita.

Le memorie ed i libri inviati in dono saranno annunziati fra le pubblicazioni recenti.

Nella fiducia di ricevere presto da V. S. una favorevole risposta, gli scriventi le offrono l'omaggio della loro alta considerazione.

Prof. FRANCESCO BRIOSCHI.
Prof. LUIGI CREMONA.

THE END OF VOL. VIII.

Printed by William Metcalfe, Green Street, Cambridge.

Fig. 1

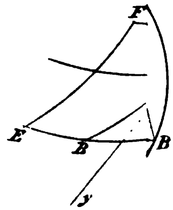
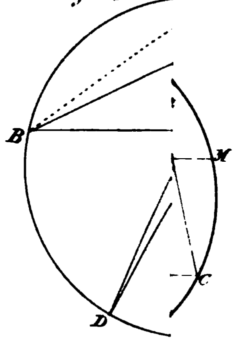
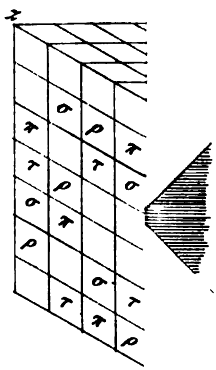


Fig. 5





Cambridge

. 12

alfe, Litho.

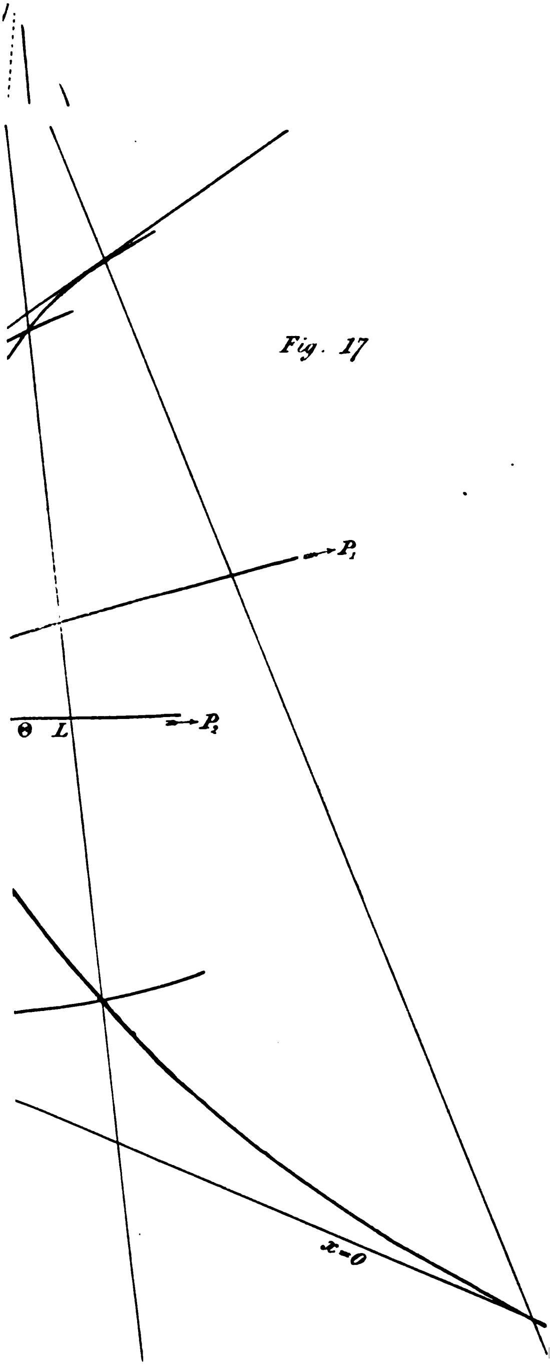


Fig. 17

W. Metcalfe, Litho Cambridge.

Fig. 22

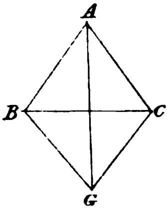


Fig. 23

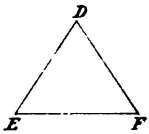
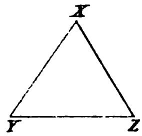


Fig. 24



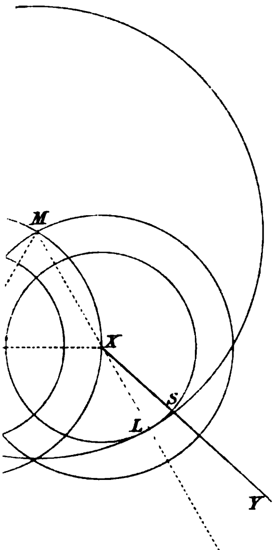


Fig. 28

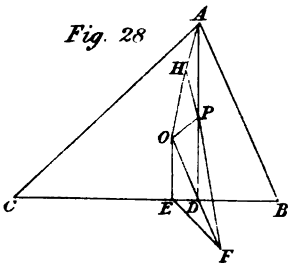


Fig. 31.